# ARBITRAGE PRICING OF SEVERAL NEW EXOTIC OPTIONS: THE PARTIAL TUNNEL AND GET-OUT OPTIONS

By

DAVID AARON BRASK

A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1999

To my brother

## ACKNOWLEDGMENTS

I would especially like to thank my family for their unconditional love and support. I thank my advisor for introducing me to this field and guiding me through my research, each of my committee members for their input and all of my friends who reminded me that social life is still a big part of life even when you are working on a Ph.D. I would also like to express my appreciation to Jim Cohen at William R. Hough & Co. for the internship and consulting opportunities through which I have gained invaluable experience.

TABLE OF CONTENTS

Abstract of Dissertation Presented to the Graduate School
of the University of Florida in Partial Fulfillment of the
Requirements for the Degree Doctor of Philosophy

ARBITRAGE PRICING OF SEVERAL NEW EXOTIC OPTIONS: THE PARTIAL TUNNEL AND GET-OUT OPTIONS

By

DAVID BRASK

December 1999

Chairman: Dr. Joseph Glover
Major Department: Mathematics

We begin with an introduction to the classic Black-Scholes option pricing model in chapter 1. This explains the method by which stochastic processes are used to obtain arbitrage prices for options. Chapter 2 follows with recent results in the field and touches on how some of these results will be extended in chapter 4.

Chapter 3 provides some new results on Brownian motion, which are of great use in chapter 4. Chapter 4 begins by introducing the first new class of options, the partial barrier tunnel options. It then proceeds to price these options using the results from chapter 3. Chapter 5 is a follow-up of chapter 4. It first shows how the results of chapter 4 generalize those in the existing literature and then provides numerical results and an analysis that illustrates how changing the parameters affects the price.

Chapter 6 introduces another new option, the Get-Out option. This option depends on two underlying securities whereas all of the aforementioned options depend only on one. The goal again is to derive an expression for the arbitrage price of this option. We separate the pricing into two cases. The first case assumes that the two underlying securities are independent and therefore do not require joint distributions for the processes. The second case, however, assumes that they are correlated. Consequently, the pricing is not as straightforward and requires joint distributions of the processes and some of their functionals.

# CHAPTER 1
# INTRODUCTION

## 1.1 Background

The financial markets are growing more rapidly now than ever before. Investors at all levels are using financial products to hedge their positions, that is, to help reduce their risk. As a result, the derivatives market has grown as well. One of the most basic and earliest financial instruments used for hedging is the standard European call option on stock. This call option gives its owner the right, but not the obligation, to purchase a security at a prespecified price and time in the future. The prespecified price is called the strike price and the prespecified time is the expiry. A put option is similar in nature but gives the holder the right to sell instead of buy. American options are very similar to European options but they allow the owner to exercise the option at any point up until expiry, whereas European options may only be exercised at expiry. From here on we will work with only European options.

Suppose one holds a long position in a particular stock. Purchasing a put option on the same stock would guarantee this investor the ability to sell the stock at the strike price. The downside loss would then be reduced. Determining prices for options similar to these but with additional stipulations is the point of this work.

## 1.2 Arbitrage Pricing

One can see that the price of a standard call option must be less than or equal to the current price of the underlying stock. If this were not the case, one could sell the call option and buy the stock. Since the option costs more, this investor would also have some cash left over. Furthermore, at expiration only two things can occur. If the call option were exercised, he would only be left with the cash left from the original transaction. If it is not exercised, then he could simply sell the stock back into the market and have even more money. In particular, if the call option is priced higher than the stock, then there are opportunities to make money without any initial investment or risk. Thus, if one were to purchase a call option, he would not pay more than the current stock price. Arguments similar to this dictate ranges in which prices must fall.

**Definition 1.2.1** *An arbitrage opportunity is a situation in which one may make a profit with no initial investment and without taking any risk.*

If two exchanges carry the same security but at different prices, then an individual could simultaneously purchase the security at the lower price and sell at the higher price. This is an example of an arbitrage opportunity. Next we will see how arbitrage can be used to price a call option.

## 1.3 The Discrete Case

### 1.3.1 The Model

We begin with the simplest model available for a stock, the single step binomial model. This model assumes that a stock either goes up or down by certain percentages over the lifetime of the option [1]. For simplicity we will assume that we have zero interest

rates here. Denote by $S_0, S_u, S_d$ and $k$ the initial price, the price if the stock goes up, the price if it goes down and the strike price.

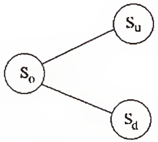


Figure 1.3: Discrete model

If the stock goes up, then the value of the call option is

$$(S_u - k)^+ = \max(S_u - k, 0). \tag{1.1}$$

If it goes down then it is worth

$$(S_d - k)^+ = \max(S_d - k, 0). \tag{1.2}$$

One could also choose to create a portfolio of the stock and a cash bond valued at 1 dollar so that the net worth of the portfolio would be the same as the value of the call option at expiry. To see how to do this we start with $x$ units of stock and $y$ units of bond. Depending upon whether the stock goes up or down, $x$ and $y$ must satisfy the equations

$$\begin{aligned} x \cdot S_u + y \cdot 1 &= (S_u - k)^+ \\ x \cdot S_d + y \cdot 1 &= (S_d - k)^+ \end{aligned} \tag{1.3}$$

in order to replicate the payoff of the call option. We can solve this system for $x$ and $y$ so that the portfolio has the same terminal value as the call option. The cost of composing

such a portfolio would then have to be the appropriate price for the call option. Any other price would lead to an arbitrage opportunity.

**Definition 1.3.1** *The arbitrage price for an option is the value of the portfolio that replicates the payoff of the option.*

1.3.2 The Martingale Measure

Notice that we did not use any probability in the preceding arbitrage arguments. However, pricing the call option in this manner does indeed induce a probability measure. Let $C$ be the price of the call option and $S_T$ be the price of the stock at expiry. We would like a probability measure $P$ under which $C$ is the expected value of the possible outcomes. That is, $P$ must satisfy

$$C = E^P\left[(S_T - k)^+\right] = p_1 \cdot (S_u - k)^+ + p_2 \cdot (S_d - k)^+ \tag{1.4}$$

where $p_1$ and $p_2$ are the probabilities of the stock going up ($S_T = S_u$) or down ($S_T = S_d$) under $P$. Since $p_1$ and $p_2$ must also satisfy $p_1 + p_2 = 1$, it is clear that there is a unique solution for $p_1$ and $p_2$. Moreover, the relationship $(S_d - k)^+ \le C \le (S_u - k)^+$ must hold in order to prevent arbitrage and this ensures that both $p_1$ and $p_2$ are nonnegative. The probability measure $P$ induced in this manner has an additional implication. We must first clarify a couple of concepts. We first introduce the more general discrete process $S = \{S_i\}_{i=0}^n$ defined on a sample space $\Omega$. The process $S$ may branch up or down, as in the single step model, at each step. The sample space $\Omega$ is composed of all possible trajectories of the process. This more general model is known as the Cox-Ross-Rubinstein model [9].

**Definition 1.3.2** *The filtration generated by the discrete process* $S = \{S_i\}_{i=0}^n$ *is the collection* $\mathcal{F}^S = \{\mathcal{F}_i\}_{i=0}^n$ *of* $\sigma$ *– algebras* $\mathcal{F}_i = \sigma\left(S_j : j \le i\right)$ *generated by the process.*

**Definition 1.3.3** *A discrete process* $S = \{S_i\}_{i=0}^n$ *is a martingale with respect to a measure* $P$ *and its filtration* $\mathcal{F}^S = \{\mathcal{F}_i\}_{i=0}^n$ *if* $E|S_i| < \infty$ *for all i and* $E\left[S_j \middle| \mathcal{F}_i\right] = S_i$ *for all* $i \le j$.

Under this measure, the stock price turns out to be a martingale. In general, the interest rates are not equal to zero. When interest rates are nonzero, we must adjust for the time value of money. In particular, receiving $x$ dollars at a future time $t$ is equivalent to purchasing $e^{-rt}x$ worth of a bond with interest rate $r$ now. Note that $e^{-rt}x$ invested in this bond will payoff $e^{rt}\left(e^{-rt}x\right) = x$ dollars at time $t$. So the value of receiving $x$ dollars at a future time $t$ is worth $e^{-rt}x$ dollars now. This is what we call a discounting factor. So in the case of nonzero interest rates, the discounted stock price process, $e^{-rt_i}S_{t_i}$, will be a martingale under this measure.

**Definition 1.3.4** *A martingale measure is a measure under which the discounted stock price is a martingale.*

Once we have a martingale measure, we may price other options by simply taking the expected value of their discounted payoff under this measure.

**Theorem 1.3.4** *The arbitrage price* $C$ *for an option is given by* $C = E^P\left[X\right]$ *where* $P$ *is the martingale measure and* $X$ *is the discounted payoff of the option.*

We omit the proof of this theorem as it may be found in [9]. This reduces the problem of pricing options to finding the martingale measure. Once the martingale measure is found, we may simply take the expected value of the discounted payoff to obtain the arbitrage price of any option.

## 1.4 The Continuous Case

### 1.4.1 The Black-Scholes Model

We now move to a better model that allows for more than just two possible outcomes for the stock price. In fact, this model allows the stock price to take any positive value. A stochastic process is used to model the stock price and an exponential function (deterministic) is used for the bond price [1].

The Black-Scholes model for options pricing is the most widely accepted and used model in the field [3]. Despite some of its unrealistic assumptions such as continuous time trading and no transaction costs, the model has persisted in both academia and industry since its birth in 1973. We first fix a probability space $(\Omega, \mathcal{F}, P)$ on which we define a standard one-dimensional Brownian motion $W_t$. Denote by $\mathcal{F}_t = \sigma(W_s; 0 \le s \le t)$ the $\sigma -$algebra generated by $W_t$. The basic Black-Scholes model is set forth by the following stochastic differential equations

$$\begin{aligned} dS_t &= \sigma S_t dW_t + \mu S_t dt \\ dB_t &= r B_t dt \end{aligned} \tag{1.5}$$

where $S_t$ denotes the stock price process, $B_t$ the bond price and $\sigma, \mu$ and $r$ are constants representing the volatility, drift and interest rate, respectively. Corresponding to each

option, there is a payoff that is dependent upon what the underlying security does by the expiry time $T$.

**Definition 1.4.1** *A contingent claim* $X$ *is an* $\mathcal{F}_T$ *measurable random variable.*

A contingent claim is used to describe the payoff of an option. Thus the contingent claim for a standard European call option is $X = (S_T - k)^+ \in \mathcal{F}_T$.

1.4.2 The Martingale Measure

Just as in the discrete case, we need only find a martingale measure under which we can compute option prices by simply taking the expected values. Though not as simple as solving a system of linear equations as in the discrete case, we are guaranteed a martingale measure by the well-known Cameron-Martin-Girsanov theorem [7]. Under this measure the discounted stock price process is a martingale. In other words, if we let $Z_t = \frac{S_t}{B_t}$ be the discounted stock price process, then we have $dZ_t = \sigma Z_t d\tilde{W}_t$ where $\tilde{W}_t$ is a Brownian motion under the martingale measure. Thus we may price options by taking their expected value under this measure. Since this measure always exists in our setting, we will assume that the measure $P$ we use to price options is the martingale measure. Furthermore, we will drop the $\tilde{W}_t$ notation and use $W_t$ as a Brownian motion under the martingale measure $P$. An important consequence of using the martingale measure is that the original drift term $\mu$ drops out. In order for the discounted stock price process to be a martingale, the drift term in the process must be $r - \frac{\sigma^2}{2}$ [7]. Since the original drift term is irrelevant, we choose to start with the martingale measure and avoid it altogether.

1.4.3 An Example

Take a standard European call option on a stock with volatility $\sigma$, drift $\mu$. Let $k$ and $T$ be the strike price and expiry, respectively. Let $P$ be the martingale measure. Under this measure we have

$$dS_t = \sigma S_t dW_t + r\, S_t dt \tag{1.6}$$

where $W_t$ is a Brownian motion under $P$. Note that the drift $\mu$ is irrelevant since we are using the martingale measure. The solution to this stochastic differential equation is given by

$$S_t = S_0 \exp\left(\sigma W_t + \left(r - \frac{\sigma^2}{2}\right)t\right). \tag{1.7}$$

We need only compute $C = E^P\left[e^{-rT}\left(S_T - k\right)^+\right]$. We use the superscript to emphasize that this expected value is taken under the martingale measure $P$. This expression depends only on the marginal distribution of $S_T$. After making appropriate substitutions we may express the price $C$ in terms of the normal cumulative distribution function as

$$C = S_0 N\left(\frac{\ln\left(\frac{S_0}{k}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - ke^{-rT} N\left(\frac{\ln\left(\frac{S_0}{k}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \tag{1.8}$$

where $N(\cdot)$ is the normal cumulative distribution function (i.e. $N(x) = P(X \le x)$ where $X$ is a standard normal random variable). Again notice that the drift term $\mu$ does not appear in this formula.

## CHAPTER 2
## RECENT RESULTS

### 2.1 Barrier Options

One of the first modifications that can be made to the basic call or put option is the addition of a barrier. A barrier is a prespecified price level, which is monitored in order to see whether or not the underlying stock hits the barrier during the course of the option. These new options, called barrier options, come in two types: knock-out and knock-in [9]. A knock-out barrier option would be the same as a regular option, but with the added stipulation that the stock price must not hit the barrier. If the stock price hits the barrier during the course of the option, then the option automatically expires with zero payoff.

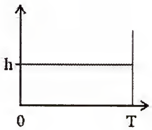


Figure 2.1: Barrier option

On the other hand, a knock-in option has zero payoff by default. The option's payoff takes place only if the stock price hits the barrier during the lifetime of the option. For example, the payoff of a knock-out barrier call option could be of the form

$$1_{\{S_t \neq h \, \forall t \in [0,T]\}} (S_T - k)^+ \qquad (2.1)$$

where $h$ is the barrier level. In this example the barrier is placed below the initial stock price, that is $h < S_0$. This type is commonly referred to as a down and out option. Pricing such an option depends not only on the marginal distribution of $S_T$, but also on whether or not the stock price hits the barrier. Depending upon whether the barrier is below or above the current stock price, the joint density for the pair $(S_T, m_T)$ or $(S_T, M_T)$ is necessary [12]. Here $m_t$ and $M_t$ denote the running minimum and maximum of the process $W_t$ as given by

$$\begin{aligned} m_t &= \inf_{0<s<t} W_s \\ M_t &= \sup_{0<s<t} W_s \end{aligned} \tag{2.2}$$

These densities are readily found in most texts on stochastic processes including [7] and [10]. The density is given by

$$P\left(W_t \in dx, M_t \in dy\right) = \frac{2(2y-x)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2y-x)^2}{2t}\right) dx\, dy . \tag{2.3}$$

The barrier option price admits a representation by a linear combination of cumulative normal density functions [9].

Barrier options have a variety of applications in the markets. Consider the case where one has a future cash flow in another currency [2]. The exchange rate affects the value of this cash flow relative to the investor. A down and out call option on the exchange rate would enable him to hedge his position. If the exchange rate fell sharply, then the call option would be of little use anymore and the option get knocked out. Thus the introduction of the barrier would make the call option less expensive for the investor and still suit his needs.

## 2.2 Partial Barrier Options

A further generalization of the barrier option is the partial barrier [4]. The only difference between a partial barrier option and a regular barrier option is that the barrier is not monitored for the entire lifetime of the option. The monitoring period would either start at time zero and end at some time before expiry or it would begin at some time before expiry and end at the expiry. Notice that for the latter case, the boundary could be hit from above or below.

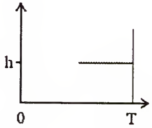


Figure 2.2: Partial barrier option

These two cases are distinguished and priced separately in [4]. The payoff for a partial barrier option where the monitoring period starts at time zero is

$$1_{\{S_t \neq h \forall t \in [0,t_1]\}} (S_T - k)^+ \tag{2.4}$$

where $t_1 < T$. Conditional expectations and the Markov property are used in [4] to compute the expectation of each payoff. The prices are expressed as linear combinations of bivariate and univariate normal cumulative distribution functions. The bivariate cumulative distribution function is defined as $N(X,Y,\rho) = P(X < x, Y < y)$ where $X$ and $Y$ are normal random variables with correlation coefficient $\rho$. A more direct

approach for several of the computations above is found in [2]. The joint density for the pair $(B_t, M_s)$ where $s \neq t$ is derived and then used to price a knock-out call option where the partial barrier begins at time zero. Not surprisingly, the joint density is a linear combination of bivariate and univariate normal density functions.

Partial barrier options may be of use both individually or when packaged with other options. The nature of this option allows one to compensate for changes in volatility over the lifetime of the option. One may expect moderate volatility in the near future but increased volatility soon after. In this case, one could choose a partial barrier option with the barrier only monitored in the near future. This would serve to decrease the cost of the option and still serve to hedge or speculate an underlying security. One may also package barrier and partial barrier options together to create ladder options [4]. A ladder option is just like a regular call or put option except for that it locks in intrinsic value when certain barriers are reached.

## 2.3 Double Barrier Options

Another modification made to the standard barrier option is the addition of a second barrier. Naturally, the two barriers are on opposite sides of the stock price. Although standard and partial barrier options use flat barriers, double barrier options where the barriers may be curved are treated in [8]. In order to price such an option, the density must reflect that the stock has not hit either of the barriers. They use an extension of Levy's density for Brownian motion confined to an interval [a,b] (i.e. a Brownian motion for which $a \leq m_t \leq M_t \leq b$). This density is based on repeated applications of the

reflection principle of Brownian motion and is written as an infinite sum [10]. The density is

$$P(A)=\int_I \frac{1}{\sqrt{2\pi t}}\sum_{j=-\infty}^{+\infty}\left(e^{-\frac{(x+2j(b-a))^2}{2t}}-e^{-\frac{(x+2j(b-a)-2b)^2}{2t}}\right)dx \tag{2.5}$$

where $A=\{a \le m_t \le M_t \le b, B_t \in I \subseteq [a,b]\}$. Their results indicate that the convergence of this series is sufficiently rapid to make it amenable to numerical implementations. Despite the generality of their results with curved boundaries, we will confine ourselves to double barrier options in which the barriers are flat. In what follows, we will refer to these types of barrier options as tunnel options.

Just like other barrier options, double barrier options may be used to speculate on the perceived volatility for an asset. There are many different ways to compute volatility, none of which are universally accepted [5]. Consider an investor who expects the volatility to be less from implied volatility of a given call option. He could sell the call option and purchase the same option with an appropriate double barrier. If he is right, then both options will be worth the same at expiry and will thus leave him with no obligations. He will profit by the amount equal to the difference in the options' prices. Double barrier options have been developed and traded by several Tokyo banks including Fuji Bank, Sanwa Bank and Nippon Credit Bank [8].

## CHAPTER 3
## SOME NEW RESULTS ON BROWNIAN MOTION

This chapter presents some new results on several functionals of Brownian motion. These results will be of great use in chapter 4. The corollaries following the main theorems provide densities for a Brownian motion at time $T$ confined to a tunnel during a fixed period. The fixed period will either start at time zero and end at some time $t$ before time $T$ or will begin at some time $t$ and end at time $T$. We use several lemmas to conceal tedious computations and obstacles in the proofs of main theorems.

**Lemma 3.1** Fix the values $x_1, x_2$ and $y_1$ and define the function $h(c,d)$ by

$$h(c,d) = \frac{1}{2\pi\sqrt{t(T-t)}} \int_{x_1}^{x_2} \int_{y_1}^{+\infty} \exp\left( cy - \frac{(x+d)^2}{2t} - \frac{(y-x)^2}{2(T-t)} \right) dy\, dx. \tag{3.1}$$

Then

$$h(c,d) = \exp\left(\frac{c^2T}{2} - cd\right)\left[ N_\rho\left(\frac{\alpha - x_1}{\sqrt{t}}, \frac{\beta - y_1}{\sqrt{T}}\right) - N_\rho\left(\frac{\alpha - x_2}{\sqrt{t}}, \frac{\beta - y_1}{\sqrt{T}}\right)\right] \tag{3.2}$$

*where $N_\rho(\cdot,\cdot)$ denotes the bivariate normal distribution function with correlation coefficient $\rho = \sqrt{\frac{t}{T}}$, $\alpha = ct - d$ and $\beta = cT - d$.*

**Proof:** Let $f(x,y) = cy - \frac{(x+d)^2}{2t} - \frac{(y-x)^2}{2(T-t)}$. We first factor out the quantity $\frac{-T}{2(T-t)}$ from $f$ so that we have

$$f(x,y)=\frac{-T}{2(T-t)}\left[-\frac{2(T-t)}{T}cy+\frac{2(T-t)}{T}\frac{(y-x)^2}{2(T-t)}+\frac{2(T-t)}{T}\frac{(x+d)^2}{2t}\right]. \tag{3.3}$$

Expansion, cancellation, grouping like terms and simplification yields

$$f(x,y)=\frac{-T}{2(T-t)}\left[\frac{x^2+2d\left(1-\frac{t}{T}\right)x}{t}-\frac{2xy}{T}+\frac{y^2-2c(T-t)y}{T}\right]-\frac{d^2}{2t}. \tag{3.4}$$

We would now like to express the quantity

$$\frac{x^2+2d\left(1-\frac{t}{T}\right)x}{t}-\frac{2xy}{T}+\frac{y^2-2c(T-t)y}{T} \tag{3.5}$$

in the form

$$\frac{(x-\alpha)^2}{t}-\frac{2(x-\alpha)(y-\beta)}{T}+\frac{(y-\beta)^2}{T}+\gamma. \tag{3.6}$$

In order to solve for the constants $\alpha$, $\beta$ and $\gamma$ we first equate the coefficients of $x$ and $y$ terms from (3.5) and (3.6). This leads to the system of equations

$$\begin{gathered}\frac{2d\left(1-\frac{t}{T}\right)}{t}=-\frac{2\alpha}{t}+\frac{2\beta}{T} \\ \frac{-2c(T-t)}{T}=-\frac{2\beta}{T}+\frac{2\alpha}{T}\end{gathered}. \tag{3.7}$$

Solving this system yields

$$\begin{aligned}\alpha&=ct-d\\ \beta&=cT-d\end{aligned}. \tag{3.8}$$

We next set the constants from both expressions equal and obtain

$$-\gamma=\frac{\alpha^2}{t}-\frac{2\alpha\beta}{T}+\frac{\beta^2}{T}. \tag{3.9}$$

So we may now rewrite (3.4) as

$$f(x,y)=\frac{-T}{2(T-t)}\left[\frac{(x-\alpha)^2}{t}-\frac{2(x-\alpha)(y-\beta)}{T}+\frac{(y-\beta)^2}{T}+\gamma\right]-\frac{c^2}{2t} \tag{3.10}$$

where $\alpha, \beta$ and $\gamma$ are defined as in (3.8) and (3.9). We take the $\gamma$ term outside of the brackets so that

$$f(x,y)=\frac{-T}{2(T-t)}\left[\frac{(x-\alpha)^2}{t}-\frac{2(x-\alpha)(y-\beta)}{T}+\frac{(y-\beta)^2}{T}\right]+\frac{-T}{2(T-t)}\gamma-\frac{d^2}{2t} \tag{3.11}$$

We now simplify the term $\frac{-T}{2(T-t)}\gamma-\frac{d^2}{2t}$. Substituting the values for $\alpha$ and $\beta$ from (3.8) into (3.9) for $\gamma$, the above expression becomes

$$\begin{aligned}\frac{-T}{2(T-t)}\gamma-\frac{c^2}{2t}&=\frac{T}{2(T-t)}\left(\frac{\alpha^2}{t}-\frac{2\alpha\beta}{T}+\frac{\beta^2}{T}\right)-\frac{c^2}{2t}\\ &=\frac{T}{2(T-t)}\left(\frac{(ct-d)^2}{t}-\frac{2(ct-d)(cT-d)}{T}+\frac{(cT-d)^2}{T}\right)-\frac{d^2}{2t}.\end{aligned} \tag{3.12}$$

After expanding and simplifying (3.12) we have

$$\frac{-T}{2(T-t)}\gamma-\frac{d^2}{2t}=\frac{c^2T}{2}-cd\,. \tag{3.13}$$

Substituting (3.13) into (3.11) yields

$$f(x,y)=\frac{-T}{2(T-t)}\left[\frac{(x-\alpha)^2}{t}-\frac{2(x-\alpha)(y-\beta)}{T}+\frac{(y-\beta)^2}{T}\right]+\frac{c^2T}{2}-cd\,. \tag{3.14}$$

Set $\sigma_x=\sqrt{t}$, $\sigma_y=\sqrt{T}$ and $\rho=\sqrt{\frac{t}{T}}$. Then we have $\frac{-1}{\sqrt{1-\rho^2}}=\frac{-T}{2(T-t)}$ and $\frac{\rho}{\sigma_x\sigma_y}=\frac{1}{T}$

so that (3.14) becomes

$$f(x,y)=\frac{-1}{2\sqrt{1-\rho^2}}\left[\frac{(x-\alpha)^2}{\sigma_x{}^2}-\frac{2\rho(x-\alpha)(y-\beta)}{\sigma_x\sigma_y}+\frac{(y-\beta)^2}{\sigma_y{}^2}\right]+\frac{c^2T}{2}-cd. \tag{3.15}$$

Recall that $h(c,d)=\frac{1}{2\pi\sqrt{t(T-t)}}\int_{x_1}^{x_2}\int_{y_1}^{+\infty}\exp\left(cy-\frac{(x+d)^2}{2t}-\frac{(y-x)^2}{2(T-t)}\right)dy\,dx$. Thus we have

$$\begin{aligned}h(c,d)&=\frac{1}{2\pi\sqrt{t(T-t)}}\int_{x_1}^{x_2}\int_{y_1}^{+\infty}\exp(f(x,y))\,dy\,dx\\&=\frac{e^{\left(\frac{b^2T}{2}-bc\right)}}{2\pi\sqrt{t(T-t)}}\int_{x_1}^{x_2}\int_{y_1}^{+\infty}e^{\frac{-1}{2\sqrt{1-\rho^2}}\left[\frac{(x-\alpha)^2}{\sigma_x{}^2}-\frac{2\rho(x-\alpha)(y-\beta)}{\sigma_x\sigma_y}+\frac{(y-\beta)^2}{\sigma_y{}^2}\right]}dy\,dx\end{aligned} \tag{3.16}$$

Making the substitutions $\tilde{x}=\frac{\alpha-x}{\sigma_x}$, $\tilde{y}=\frac{\beta-y}{\sigma_y}$ and switching the limits of integration yields (3.2). □

**Lemma 3.2** *The density*

$$k(x)=\frac{1}{\sqrt{2\pi t}}\sum_{j=-\infty}^{+\infty}\left[\exp\left[-\frac{(x+2j(b-a))^2}{2t}\right]-\exp\left[-\frac{(x+2j(b-a)-2b)^2}{2t}\right]\right]$$ *is uniformly convergent on the interval* $[a,b]$.

**Proof:** If a series of the form

$$\sum_{j=-\infty}^{+\infty}\exp\left[-\frac{(x+2j(b-a))^2}{2t}\right] \tag{3.17}$$

converges uniformly on the interval $[a,b]$, then a series of the form

$$\sum_{j=-\infty}^{+\infty}\exp\left[-\frac{(x+2j(b-a)-2b)^2}{2t}\right] \tag{3.18}$$

will also converge uniformly on the same interval. To see this, first recall that $a < 0 < b$.

Then observe that

$$-\frac{(x+2j(b-a))^2}{2t} > -\frac{(x+2(j+1)(b-a)-2b)^2}{2t}$$

for large positive values of $j$ and

$$-\frac{(x+2j(b-a))^2}{2t} > -\frac{(x+2j(b-a)-2b)^2}{2t}$$

for large negative values of $j$. So the convergence of (3.17) ensures that (3.18) will also converge. Thus we need only show that a series of the form (3.17) converges uniformly on the interval $[a,b]$. However, this is equivalent to the one sided series

$$\sum_{j=1}^{+\infty} \exp\left[-\frac{(x+2j(b-a))^2}{2t}\right] \tag{3.19}$$

converging uniformly on the same interval. Now we choose an integer $N$ large enough that $j \geq N$ implies $x + 2j(b-a) > 0$. We then observe that for $j \geq N$ we have

$$\exp\left[-\frac{(x+2j(b-a))^2}{2t}\right] \leq \exp\left[-\frac{(a+2j(b-a))^2}{2t}\right] \qquad \forall x \in [a,b]. \tag{3.20}$$

Now the series

$$\sum_{j=1}^{+\infty} \exp\left[-\frac{(a+2j(b-a))^2}{2t}\right] \tag{3.21}$$

clearly converges. If we define

$$M_n = \sup_{x\in[a,b]} \left( \sum_{j=1}^{+\infty} \exp\left[-\frac{(x+2j(b-a))^2}{2t}\right]\right), \tag{3.22}$$

then we have

$$\begin{aligned}\lim_{n\to+\infty} M_n &= \lim_{n\to+\infty} \sup_{x\in[a,b]}\left(\sum_{j=n}^{+\infty}\exp\left[-\frac{(a+2j(b-a))^2}{2t}\right]\right)\\ &\le \lim_{n\to+\infty}\left(\sum_{j=n}^{+\infty}\exp\left[-\frac{(a+2j(b-a))^2}{2t}\right]\right)\\ &=0\end{aligned} \tag{3.23}$$

The inequality in (3.23) is due to (3.20). The limit goes to zero since the series converges. (3.23) is a necessary and sufficient condition for the uniform convergence of (3.19) [11]. This proves the lemma. □

**Theorem 3.3** *Let* $A=\{a\le m_t \le M_t \le b, B_T \ge y_1\}$ *where* $t<T$. *Then we have*

$$E\left[e^{\lambda B_T}\cdot 1_A\right]=\sum_{j=-\infty}^{+\infty}\left(h(\lambda,c_j)-h(\lambda,d_j)\right) \tag{3.24}$$

*where* $h(\cdot,\cdot)$ *is given by* (3.2), $c_j = 2j(b-a)$ *and* $d_j = 2j(b-a)-2b$.

**Proof:** Let $A_1=\{a\le m_t\le M_t\le b\}$ and $A_2=\{B_T\ge y_1\}$ so that $A=A_1\cap A_2$. Then we have

$$E\left[e^{\lambda B_T}\cdot 1_A\right]=E\left[e^{\lambda B_T}\cdot 1_{A_1}1_{A_2}\right]. \tag{3.25}$$

Using a simple property of expectations, we take an inner expectation conditioned on $\mathcal{F}_t$ so that (3.25) becomes

$$E\left[E\left[e^{\lambda B_T}\cdot 1_{A_1}1_{A_2}\middle|\mathcal{F}_t\right]\right]. \tag{3.26}$$

We now note that the random variable $1_{A_1}$ is $\mathcal{F}_t$-measurable. Thus we can take $1_{A_1}$ outside of the inner conditional expectation so that (3.26) becomes

$$E\left[1_{A_1}\cdot E\left[e^{\lambda B_T}\cdot 1_{A_2}\middle|\mathcal{F}_t\right]\right]. \tag{3.27}$$

Next we rewrite (3.27) using the shift operator $\theta_t$ and the Markov property of Brownian motion. This yields

$$E\left[1_{A_1}\cdot E\left[e^{\lambda B_T}\cdot 1_{A_2}\middle|\mathcal{F}_t\right]\right]=E\left[1_{A_1}\cdot E\left[\left(e^{\lambda B_{T-t}}\cdot 1_{\tilde{A}_2}\right)\circ\theta_t\middle|\mathcal{F}_t\right]\right]$$
$$=E\left[1_{A_1}\cdot E^{B_t}\left[e^{\lambda B_{T-t}}\cdot 1_{\tilde{A}_2}\right]\right] \tag{3.28}$$

where $\tilde{A}_2=\{B_{T-t}\geq y_1\}$ and $E^z[\cdot]$ denotes the expectation for the process started at $z$ instead of the origin. We may compute the inner expectation as

$$E^{B_t}\left[e^{\lambda B_{T-t}}\cdot 1_{\tilde{A}_2}\right]=\frac{1}{\sqrt{2\pi(T-t)}}\int_{y_1}^{+\infty}\exp(\lambda y)\cdot\exp\left(-\frac{(y-B_t)^2}{2(T-t)}\right)dy, \tag{3.29}$$

which is a function of $B_t$. So we define the function $f$ by

$$f(x)=E^x\left[e^{\lambda B_{T-t}}\cdot 1_{\tilde{A}_2}\right]=\frac{1}{\sqrt{2\pi(T-t)}}\int_{y_1}^{+\infty}\exp(\lambda y)\cdot\exp\left(-\frac{(y-x)^2}{2(T-t)}\right)dy \tag{3.30}$$

and write (3.28) as

$$E\left[1_{A_1}\cdot f(B_t)\right]. \tag{3.31}$$

Recall that $A_1=\{a\leq m_t\leq M_t\leq b\}$. So we compute (3.31) as

$$E\left[1_{A_1}\cdot f(B_t)\right]=\int_a^b k(x)\cdot f(x)\,dx \tag{3.32}$$

where

$$k(x)=\frac{1}{\sqrt{2\pi t}}\sum_{j=-\infty}^{+\infty}\left[\exp\left[-\frac{(x+2j(b-a))^2}{2t}\right]-\exp\left[-\frac{(x+2j(b-a)-2b)^2}{2t}\right]\right]. \tag{3.33}$$

Set $c_j=2j(b-a)$ and $d_j=2j(b-a)-2b$. Then we may express (3.32) as

$$\int_a^b\left(\frac{1}{\sqrt{2\pi t}}\sum_{j=-\infty}^{+\infty}\left[\exp\left[-\frac{(x+c_j)^2}{2t}\right]-\exp\left[-\frac{(x+d_j)^2}{2t}\right]\right]\right)\cdot f(x)dx. \tag{3.34}$$

Substituting $f(x)$ back into (3.34) and pulling the constants to the front of the summation we obtain

$$\begin{aligned}&\int_a^b\left(\frac{1}{2\pi\sqrt{t(T-t)}}\sum_{j=-\infty}^{+\infty}\left[\exp\left[-\frac{(x+c_j)^2}{2t}\right]-\exp\left[-\frac{(x+d_j)^2}{2t}\right]\right]\right)\times\\ &\qquad\int_I\exp(\lambda y)\cdot\exp\left(-\frac{(y-x)^2}{2(T-t)}\right)dy\,dx\end{aligned} \tag{3.35}$$

where $I=[y_1,\infty)$. Since $f(x)$ is bounded on the interval $[a,b]$, we use lemma 3.2 to interchange the infinite limit with the outer integral. We may also combine the integrands to obtain

$$\begin{aligned}&\sum_{j=-\infty}^{+\infty}\left[\frac{1}{2\pi\sqrt{t(T-t)}}\int_a^b\int_I\left(\exp\left[-\frac{(x+c_j)^2}{2t}\right]-\exp\left[-\frac{(x+d_j)^2}{2t}\right]\right)\times\right.\\ &\qquad\left.\left(\exp(\lambda y)\cdot\exp\left(-\frac{(y-x)^2}{2(T-t)}\right)\right)dy\,dx\right].\end{aligned} \tag{3.36}$$

Distributing the exponentials on the right yields

$$\begin{aligned}&\sum_{j=-\infty}^{+\infty}\left[\frac{1}{2\pi\sqrt{t(T-t)}}\int_a^b\int_I\left(\exp\left[-\frac{(x+c_j)^2}{2t}+\lambda y-\frac{(y-x)^2}{2(T-t)}\right]-\right.\right.\\ &\qquad\left.\left.\exp\left[-\frac{(x+d_j)^2}{2t}+\lambda y-\frac{(y-x)^2}{2(T-t)}\right]\right)dy\,dx\right].\end{aligned} \tag{3.37}$$

We now have a doubly infinite series where the terms are of the form

$$h(\lambda,c_j)-h(\lambda,d_j) \tag{3.38}$$

where $h(\cdot,\cdot)$ is defined as in (3.2). We use lemma 3.1 to complete the proof. □

**Corollary 3.4** $P\left(a \le m_t \le M_t \le b, B_T \ge y_1\right) = \sum_{j=-\infty}^{+\infty} \left(h(0,c_j) - h(0,d_j)\right)$ for $t < T$.

**Proof:** This is immediate from theorem 3.3. □

Again we fix the constants $x_1, x_2$ and $y_1$ and define a new function $\hat{h}(c,d)$ by

$$\hat{h}(c,d) = \frac{1}{2\pi\sqrt{t(T-t)}} \int_{x_1}^{x_2} \int_{-\infty}^{y_1} \exp\left(cy - \frac{(x+d)^2}{2t} - \frac{(y-x)^2}{2(T-t)}\right) dy\, dx. \tag{3.39}$$

**Remark:** *This function differs from $h(c,d)$ only by the limits of integration on the y variable.*

**Lemma 3.5** *The function $\hat{h}(c,d)$ may be expressed as*

$$\hat{h}(c,d) = \exp\left(\frac{c^2 T}{2} - cd\right)\left[N_\rho\left(\frac{x_2-\alpha}{\sqrt{t}}, \frac{y_1-\beta}{\sqrt{T}}\right) - N_\rho\left(\frac{x_1-\alpha}{\sqrt{t}}, \frac{y_1-\beta}{\sqrt{T}}\right)\right] \tag{3.40}$$

*where $\rho = \sqrt{\frac{t}{T}}$, $\alpha = ct - d$ and $\beta = cT - d$.*

**Proof:** This proof is the same as the proof of lemma 3.1 except for making the last substitutions

$$\tilde{x} = \frac{x-\alpha}{\sigma_x} \text{ and } \tilde{y} = \frac{y-\beta}{\sigma_y} \tag{3.41}$$

instead of

$$\tilde{x} = \frac{\alpha - x}{\sigma_x} \text{ and } \tilde{y} = \frac{\beta - y}{\sigma_y}. \tag{3.42}$$

□

**Lemma 3.6** Define the function $g(c,d)$ by

$$g(c,d)=\frac{1}{2\pi\sqrt{t(T-t)}}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\exp\left(cy-\frac{(x+d)^2}{2t}-\frac{(y-x)^2}{2(T-t)}\right)dy\,dx. \tag{3.43}$$

Then

$$g(c,d)=\exp\left(\frac{c^2T}{2}-cd\right)\left[\left\{N_\rho\left(\frac{x_2-\alpha}{\sqrt{t}},\frac{y_2-\beta}{\sqrt{T}}\right)-N_\rho\left(\frac{x_1-\alpha}{\sqrt{t}},\frac{y_2-\beta}{\sqrt{T}}\right)\right\}-\left\{N_\rho\left(\frac{x_2-\alpha}{\sqrt{t}},\frac{y_1-\beta}{\sqrt{T}}\right)-N_\rho\left(\frac{x_1-\alpha}{\sqrt{t}},\frac{y_1-\beta}{\sqrt{T}}\right)\right\}\right] \tag{3.44}$$

*where* $N_\rho(\cdot,\cdot)$ *denotes the bivariate normal distribution function with correlation*

*coefficient* $\rho=\sqrt{\frac{t}{T}}$, $\alpha=ct-d$ *and* $\beta=cT-d$.

**Proof:** This proof is immediate since $g(c,d)$ is the difference of two functions having the

form $h(c,d)$ or $\hat{h}(c,d)$. In particular, any integral of the form $\int_{x_1}^{x_2}\int_{y_1}^{y_2}(\cdot)\,dy\,dx$ may be

written as $\int_{x_1}^{x_2}\int_{-\infty}^{y_2}(\cdot)\,dy\,dx-\int_{x_1}^{x_2}\int_{-\infty}^{y_1}(\cdot)\,dy\,dx$ so that the right hand side is the difference of two

functions of the form $\hat{h}(c,d)$. □

**Theorem 3.7** *Define* $w(c,d)$ *by*

$$w(c,d)=\exp\left(\frac{c^2T}{2}-cd\right)\times\left[\left\{N_\rho\left(\frac{x_2+d-\alpha}{\sqrt{t}},\frac{y_2-\beta}{\sqrt{T}}\right)-N_\rho\left(\frac{x_1+d-\alpha}{\sqrt{t}},\frac{y_2-\beta}{\sqrt{T}}\right)\right\}-\left\{N_\rho\left(\frac{x_2+d-\alpha}{\sqrt{t}},\frac{y_1-\beta}{\sqrt{T}}\right)-N_\rho\left(\frac{x_1+d-\alpha}{\sqrt{t}},\frac{y_1-\beta}{\sqrt{T}}\right)\right\}\right] \tag{3.45}$$

*and set* $B = \{a \le B_u \le b \ \forall u \in [t,T], B_T \in I\}$ *where* $t < T$ *and* $I = [y_1, y_2] \subset [a,b]$. *Then we have*

$$E\left[e^{\lambda B_T} \cdot 1_B\right] = \sum_{j=-\infty}^{+\infty} \left(w(\lambda, c_j) - w(\lambda, d_j)\right) \tag{3.46}$$

*where* $c_j = 2j(b-a)$ *and* $d_j = 2j(b-a) - 2b$.

**Proof:** Just as in theorem 3.3 we use basic properties of conditional expectation and the Markov property of Brownian motion to obtain

$$\begin{aligned} E\left[e^{\lambda B_T} \cdot 1_B\right] &= E\left[E\left[e^{\lambda B_T} \cdot 1_B \middle| \mathcal{F}_t\right]\right] \\ &= E\left[E\left[\left(e^{\lambda B_{T-t}} \cdot 1_{\tilde{B}}\right) \circ \theta_t \middle| \mathcal{F}_t\right]\right] \\ &= E\left[E^{B_t}\left[e^{\lambda B_{T-t}} \cdot 1_{\tilde{B}}\right]\right] \end{aligned} \tag{3.47}$$

where $\tilde{B} = \{a \le B_u \le b \ \forall u \in [0, T-t], B_{T-t} \in I\}$. We may compute the inner expectation as

$$E^{B_t}\left[e^{\lambda B_{T-t}} \cdot 1_{\tilde{B}}\right] = \int_I k(y - B_t) \cdot \exp(\lambda y)\, dy \tag{3.48}$$

where

$$k(x) = \frac{1}{\sqrt{2\pi(T-t)}} \sum_{j=-\infty}^{+\infty} \left[\exp\left[-\frac{(x + 2j(b-a))^2}{2(T-t)}\right] - \exp\left[-\frac{(x + 2j(b-a) - 2b)^2}{2(T-t)}\right]\right]. \tag{3.49}$$

Define the function $f(x)$ by

$$f(x) = E^x\left[e^{\lambda B_{T-t}} \cdot 1_{\tilde{B}}\right] = \int_I k(y - x) \cdot \exp(\lambda y)\, dy\,. \tag{3.50}$$

We may then compute the entire expectation from (3.47) as

$$E\left[E^{B_t}\left[e^{\lambda B_{T-t}}\cdot 1_{\tilde{B}}\right]\right]=E[f(B_t)] \\ =\frac{1}{\sqrt{2\pi t}}\int_a^b f(x)\cdot\exp\left(-\frac{x^2}{2t}\right)dx \tag{3.51}$$

We now substitute the function $f(x)$ into (3.51) to obtain

$$\frac{1}{\sqrt{2\pi t}}\int_a^b\int_I k(y-x)\cdot\exp(\lambda y)\,dy\cdot\exp\left(-\frac{x^2}{2t}\right)dx \tag{3.52}$$

and the density $k(x)$ into (3.52) which becomes

$$\frac{1}{\sqrt{2\pi t}}\int_a^b\int_I\frac{1}{\sqrt{2\pi(T-t)}}\sum_{j=-\infty}^{+\infty}\left[\exp\left(\frac{(y-x+c_j)^2}{2(T-t)}\right)-\exp\left(\frac{(y-x+d_j)^2}{2(T-t)}\right)\right]\times \\ \exp(\lambda y)\,dy\cdot\exp\left(-\frac{x^2}{2t}\right)dx \tag{3.53}$$

Now $f(x)$ is bounded so we may use the uniform convergence of the series to move the summation outside of the integral signs. After simplifying (3.53) becomes

$$\sum_{j=-\infty}^{+\infty}\frac{1}{2\pi\sqrt{t(T-t)}}\int_a^b\int_I\left[\exp\left(\lambda y-\frac{(y-x+c_j)^2}{2(T-t)}-\frac{x^2}{2t}\right)-\right. \\ \left.\exp\left(\lambda y-\frac{(y-x+d_j)^2}{2(T-t)}-\frac{x^2}{2t}\right)\right]dy\,dx \tag{3.54}$$

Define

$$\tilde{w}(c,d)=\frac{1}{2\pi\sqrt{t(T-t)}}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\exp\left(cy-\frac{x^2}{2t}-\frac{(y-x+d)^2}{2(T-t)}\right)dy\,dx \tag{3.55}$$

and recall that

$$g(c,d)=\frac{1}{2\pi\sqrt{t(T-t)}}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\exp\left(cy-\frac{(x+d)^2}{2t}-\frac{(y-x)^2}{2(T-t)}\right)dy\,dx. \tag{3.56}$$

Notice that we could translate the $x$ variable in $\tilde{w}(c,d)$ and rewrite (3.55) as

$$\tilde{w}(c,d)=\frac{1}{2\pi\sqrt{t(T-t)}}\int_{x_1-d}^{x_2-d}\int_{y_1}^{y_2}\exp\left(cy-\frac{(x+d)^2}{2t}-\frac{(y-x)^2}{2(T-t)}\right)dy\,dx. \tag{3.57}$$

We appeal to lemma 3.4 to see that $\tilde{w}(c,d)$ may be written as

$$\begin{aligned} w(c,d)=\exp&\left(\frac{c^2T}{2}-cd\right)\times \\ &\left[\left\{N_\rho\left(\frac{x_2-d-\alpha}{\sqrt{t}},\frac{y_2-\beta}{\sqrt{T}}\right)-N_\rho\left(\frac{x_1-d-\alpha}{\sqrt{t}},\frac{y_2-\beta}{\sqrt{T}}\right)\right\}-\right. \\ &\left.\left\{N_\rho\left(\frac{x_2-d-\alpha}{\sqrt{t}},\frac{y_1-\beta}{\sqrt{T}}\right)-N_\rho\left(\frac{x_1-d-\alpha}{\sqrt{t}},\frac{y_1-\beta}{\sqrt{T}}\right)\right\}\right] \end{aligned} \tag{3.58}$$

where $\rho=\sqrt{\frac{t}{T}}$, $\alpha=ct-d$ and $\beta=cT-d$. But then $\tilde{w}(c,d)=w(c,d)$ so (3.54) becomes

$$\sum_{j=-\infty}^{+\infty}\left[w(\lambda,c_j)-w(\lambda,d_j)\right]. \tag{3.59}$$

This completes the proof. □

**Corollary 3.8** $P\left(a\le B_u\le b\ \forall u\in[t,T], B_T\in I\right)=\sum_{j=-\infty}^{+\infty}\left(w(0,c_j)-w(0,d_j)\right)$ for $t<T$ and $I=[y_1,y_2]\subset[a,b]$.

**Proof:** This is immediate from theorem 3.6. □

# CHAPTER 4
# PARTIAL TUNNEL OPTIONS

## 4.1 Description

A partial tunnel option is precisely what its name indicates. An option of this type does not monitor the barrier for the entire life of the option. Just as with a partial barrier option, the monitoring period either starts at time zero and ends at a certain time $t_1$ before expiry or starts at a certain non-initial time $t_1$ and ends at expiry. We distinguish these two cases as Type I and Type II options, respectively. This chapter will give explicit pricing formulas for call and put options of both Type I and Type II in terms of cumulative bivariate distribution functions.

## 4.2 Removing the Drift

Recall from chapter 1 that the price of an option is simply the expectation of its discounted payoff with respect to the martingale measure $P$. The payoffs from each of the partial tunnel options depend on the distribution of the stock price process

$$S_t = S \cdot e^{\sigma W_t + \left[r - \frac{\sigma^2}{2}\right]t} \tag{4.1}$$

where $W_t$ is a Brownian motion under $P$. All of the densities we have derived in chapter 3 and most that are found in the literature are based on Brownian motion with no drift term. However, the Brownian motion that appears in the stock price process is always coupled with a drift term. Observe that

$$\sigma W_t + \left[ r - \frac{\sigma^2}{2} \right] t = \sigma \left( W_t + \frac{1}{\sigma} \left[ r - \frac{\sigma^2}{2} \right] t \right). \tag{4.2}$$

In particular, the process we are dealing with is Brownian motion with drift $\frac{1}{\sigma}\left[ r - \frac{\sigma^2}{2} \right]$.

In order to use densities for different functionals of Brownian motion we must remove the drift term. This can readily be done using the Cameron-Martin-Girsanov theorem as we did in chapter 1. We implicitly define the probability measure $Q$, under which the process

$$\tilde{W}_t = W_t + \frac{1}{\sigma} \left[ r - \frac{\sigma^2}{2} \right] t \tag{4.3}$$

is a Brownian motion, via the Radon-Nikodym derivative

$$\begin{aligned} \frac{dP}{dQ} &= \exp\left( \mu W_T - \frac{3}{2} \mu^2 T \right) \\ &= \exp\left( \mu\left(\tilde{W}_T - \mu t\right) - \frac{3}{2} \mu^2 T \right) \\ &= \exp\left( \mu \tilde{W}_T - \frac{1}{2} \mu^2 T \right) \end{aligned} \tag{4.4}$$

where

$$\mu = \frac{1}{\sigma} \left[ r - \frac{\sigma^2}{2} \right]. \tag{4.5}$$

Observe that the quantity $\mu$ defined in (4.5) is not the original drift for the stock price process as in chapter 1. Here $\mu$ is the drift of the stock price process under the martingale measure $P$. Now the arbitrage price for an option is found by taking the expectation under the martingale measure $P$. The expectation under the measure $P$ is not equal to the expectation under the measure $Q$ in general. However, the Radon-

Nikodym derivative serves to relate the two expectations. In particular, let $X \in \mathcal{F}_T$. Then we have

$$E^P[X] = E^Q\left[\frac{dP}{dQ}X\right]. \tag{4.6}$$

Thus the arbitrage price for an option may be computed as an expectation under $Q$ of the discounted payoff multiplied by the Radon-Nikodym derivative in (4.4). In this manner we will be able to use the densities for Brownian motion with no drift.

## 4.3 Partial Tunnel Options of Type I

### 4.3.1 Partial Tunnel Call Option of Type I

A partial tunnel call option of type I (PTCO-I) has the payoff function

$$X = 1_{\{a \le S_u \le b \ \forall u \in [0,t_1]\}}(S_T - k)^+ \tag{4.7}$$

where $k$ is the strike price and the barriers are $a$ and $b$. Thus the arbitrage price $V_1^C$ of a PTCO-I is given by

$$\begin{aligned} V_1^C &= E^P\left[e^{-rT} \cdot 1_{\{a \le S_u \le b \ \forall u \in [0,t_1]\}}(S_T - k)^+\right] \\ &= E^Q\left[\frac{dP}{dQ}e^{-rT} \cdot 1_{\{a \le S_u \le b \ \forall u \in [0,t_1]\}}(S_T - k)^+\right] \end{aligned} \tag{4.8}$$

where the second equality is from (4.6). Under the new measure $Q$ we have

$$\begin{aligned} S_t &= S \cdot e^{\sigma W_t + \left[r - \frac{\sigma^2}{2}\right]t} \\ &= S \cdot e^{\sigma \tilde{W}_t} \end{aligned} \tag{4.9}$$

Define $A = \{a \le S_u \le b \ \ \forall u \in [0,t_1]\}$. Using (4.9) we rewrite $A$ as

$$\begin{aligned} A &= \{a \le S_u \le b \ \ \forall u \in [0,t_1]\} \\ &= \{\tilde{a} \le \tilde{W}_u \le \tilde{b} \ \ \forall u \in [0,t_1]\} \end{aligned} \tag{4.10}$$

where $\tilde{a} = \dfrac{\ln\left(\dfrac{a}{S}\right)}{\sigma}$ and $\tilde{b} = \dfrac{\ln\left(\dfrac{b}{S}\right)}{\sigma}$. Substituting $\dfrac{dP}{dQ}$ from (4.4) and $A$ from (4.10) we may express the price from (4.8) as

$$V_{\mathrm{I}}^{\mathrm{C}} = E^Q\left[\exp\left(\mu\tilde{W}_T - \frac{1}{2}\mu^2 T\right)e^{-rT}\cdot 1_{\left\{\tilde{a}\le\tilde{W}_u\le\tilde{b}\ \forall u\in[0,t_1]\right\}}\left(S_T - k\right)^+\right]. \tag{4.11}$$

**Lemma 4.1** *The price* $V_{\mathrm{I}}^{\mathrm{C}}$ *of the* PTCO-I *is given by*

$$\begin{aligned} V_{\mathrm{I}}^{\mathrm{C}} = e^{-rT-\frac{1}{2}\mu^2 T}\Bigg(& S\cdot E^Q\left[e^{(\mu+\sigma)\tilde{W}_T}\cdot 1_{\left\{\tilde{a}\le\tilde{W}_u\le\tilde{b}\ \forall u\in[0,t_1],\tilde{W}_T\in[\tilde{k},\infty)\right\}}\right] \\ & - k\cdot E^Q\left[e^{\mu\tilde{W}_T}\cdot 1_{\left\{\tilde{a}\le\tilde{W}_u\le\tilde{b}\ \forall u\in[0,t_1],\tilde{W}_T\in[\tilde{k},\infty)\right\}}\right]\Bigg) \end{aligned} \tag{4.12}$$

where $\tilde{k} = \dfrac{\ln\left(\dfrac{k}{S}\right)}{\sigma}$.

**Proof:** Since

$$\begin{aligned} \left\{S_T - k \ge 0\right\} &= \left\{\tilde{W}_T \ge \tilde{k}\right\} \\ &= \left\{\tilde{W}_T \in [\tilde{k},\infty)\right\}, \end{aligned} \tag{4.13}$$

$$V_{\mathrm{I}}^{\mathrm{C}} = E^Q\left[\exp\left(\mu\tilde{W}_T - \frac{1}{2}\mu^2 T\right)e^{-rT}\cdot 1_{\left\{\tilde{a}\le\tilde{W}_u\le\tilde{b}\ \forall u\in[0,t_1]\right\}}1_{\left\{\tilde{W}_T\in[\tilde{k},\infty)\right\}}\left(S_T - k\right)\right]. \tag{4.14}$$

Next we break this up into two separate expectations so that

$$\begin{aligned} V_{\mathrm{I}}^{\mathrm{C}} &= E^Q\left[\exp\left(\mu\tilde{W}_T - \frac{1}{2}\mu^2 T\right)e^{-rT}\cdot 1_{\left\{\tilde{a}\le\tilde{W}_u\le\tilde{b}\ \forall u\in[0,t_1],\tilde{W}_T\in[\tilde{k},\infty)\right\}}S_T\right] \\ &\quad - E^Q\left[\exp\left(\mu\tilde{W}_T - \frac{1}{2}\mu^2 T\right)e^{-rT}\cdot 1_{\left\{\tilde{a}\le\tilde{W}_u\le\tilde{b}\ \forall u\in[0,t_1],\tilde{W}_T\in[\tilde{k},\infty)\right\}}k\right]. \end{aligned} \tag{4.15}$$

Replacing $S_T$ with the process in (4.9) and factoring out constants yields

$$V_{\mathrm{I}}^{\mathrm{C}} = e^{-rT-\frac{1}{2}\mu^2 T}\left( S\cdot E^{Q}\left[ e^{(\mu+\sigma)\tilde{W}_T}\cdot 1_{\left\{\tilde{a}\le \tilde{W}_u\le \tilde{b}\ \forall u\in[0,t_1],\tilde{W}_T\in[\tilde{k},\infty)\right\}}\right] - k\cdot E^{Q}\left[ e^{\mu\tilde{W}_T}\cdot 1_{\left\{\tilde{a}\le \tilde{W}_u\le \tilde{b}\ \forall u\in[0,t_1],\tilde{W}_T\in[\tilde{k},\infty)\right\}}\right]\right). \tag{4.16}$$

This proves the lemma. □

**Theorem 4.2** *The price* $V_{\mathrm{I}}^{\mathrm{C}}$ *of the* PTCO-I *is given by*

$$V_{\mathrm{I}}^{\mathrm{C}} = \sum_{j=-\infty}^{+\infty} e^{-rT-\frac{1}{2}\mu^2 T}\left[ S\cdot\left(h(\mu+\sigma,c_j)-h(\mu+\sigma,d_j)\right) - k\cdot\left(h(\mu,c_j)-h(\mu,d_j)\right)\right] \tag{4.17}$$

where

$$h(c,d)=\exp\left(\frac{c^2T}{2}-cd\right)\left[N_\rho\left(\frac{\alpha-\tilde{b}}{\sqrt{t_1}},\frac{\beta-\tilde{k}}{\sqrt{T}}\right)-N_\rho\left(\frac{\alpha-\tilde{a}}{\sqrt{t_1}},\frac{\beta-\tilde{k}}{\sqrt{T}}\right)\right], \tag{4.18}$$

$c_j = 2(\tilde{b}-\tilde{a})$, $d_j = 2(\tilde{b}-\tilde{a})-2\tilde{b}$, $\rho=\sqrt{\frac{t_1}{T}}$, $\alpha = ct_1 - d$ and $\beta = cT-d$.

**Proof:** From lemma 4.1 we have

$$V_{\mathrm{I}}^{\mathrm{C}} = e^{-rT-\frac{1}{2}\mu^2 T}\left( S\cdot E^{Q}\left[ e^{(\mu+\sigma)\tilde{W}_T}\cdot 1_{\left\{\tilde{a}\le \tilde{W}_u\le \tilde{b}\ \forall u\in[0,t_1],\tilde{W}_T\in[\tilde{k},\infty)\right\}}\right] - k\cdot E^{Q}\left[ e^{\mu\tilde{W}_T}\cdot 1_{\left\{\tilde{a}\le \tilde{W}_u\le \tilde{b}\ \forall u\in[0,t_1],\tilde{W}_T\in[\tilde{k},\infty)\right\}}\right]\right). \tag{4.19}$$

Applying theorem 3.3 to each of the expectations completes the proof. □

The price $V_{\mathrm{I}}^{\mathrm{C}}$ of the PTCO-I, as well as the price of each of the other partial tunnel options, is written out explicitly in the appendix.

4.3.2 Partial Tunnel Put Option of Type I

Computations similar to those above can be used to show that the arbitrage price $V_{\mathrm{I}}^{P}$ for the partial tunnel put option of type I (PTPO-I) can be expressed as

$$V_{\mathrm{I}}^{P}=e^{-rT-\frac{1}{2}\mu^{2}T}\left(k\cdot E^{Q}\left[e^{\mu\tilde{W}_{T}}\cdot 1_{\left\{\tilde{a}\le\tilde{W}_{u}\le\tilde{b}\ \forall u\in[0,t_{1}],\tilde{W}_{T}\in(-\infty,\tilde{k})\right\}}\right]\right.$$
$$\left.-S\cdot E^{Q}\left[e^{(\mu+\sigma)\tilde{W}_{T}}\cdot 1_{\left\{\tilde{a}\le\tilde{W}_{u}\le\tilde{b}\ \forall u\in[0,t_{1}],\tilde{W}_{T}\in(-\infty,\tilde{k})\right\}}\right]\right). \qquad (4.20)$$

**Theorem 4.4** *The price* $V_{\mathrm{I}}^{P}$ *of the* PTPO-I *is given by*

$$V_{\mathrm{I}}^{P}=\sum_{j=-\infty}^{+\infty}e^{-rT-\frac{1}{2}\mu^{2}T}\left[k\cdot\left(\hat{h}(\mu,c_{j})-\hat{h}(\mu,d_{j})\right)-S\cdot\left(\hat{h}(\mu+\sigma,c_{j})-\hat{h}(\mu+\sigma,d_{j})\right)\right]. \qquad (4.21)$$

*where*

$$\hat{h}(c,d)=\exp\left(\frac{c^{2}T}{2}-cd\right)\left[N_{\rho}\left(\frac{\tilde{b}-\alpha}{\sqrt{t_{1}}},\frac{\tilde{k}-\beta}{\sqrt{T}}\right)-N_{\rho}\left(\frac{\tilde{a}-\alpha}{\sqrt{t_{1}}},\frac{\tilde{k}-\beta}{\sqrt{T}}\right)\right], \qquad (4.22)$$

$c_{j}=2(\tilde{b}-\tilde{a})$, $d_{j}=2(\tilde{b}-\tilde{a})-2\tilde{b}$, $\rho=\sqrt{\frac{t_{1}}{T}}$, $\alpha=ct_{1}-d$ and $\beta=cT-d$.

**Proof:** We have

$$V_{\mathrm{I}}^{\mathrm{P}}=e^{-rT-\frac{1}{2}\mu^{2}T}\left(k\cdot E^{Q}\left[e^{\mu\tilde{W}_{T}}\cdot 1_{\left\{\tilde{a}\le\tilde{W}_{u}\le\tilde{b}\ \forall u\in[0,t_{1}],\tilde{W}_{T}\in(-\infty,\tilde{k}]\right\}}\right]\right.$$
$$\left.-S\cdot E^{Q}\left[e^{(\mu+\sigma)\tilde{W}_{T}}\cdot 1_{\left\{\tilde{a}\le\tilde{W}_{u}\le\tilde{b}\ \forall u\in[0,t_{1}],\tilde{W}_{T}\in(-\infty,\tilde{k}]\right\}}\right]\right). \qquad (4.23)$$

Using the same methods as in proof of theorem 3.3 we can see that

$$E^{Q}\left[e^{\lambda B_{T}}\cdot 1_{A}\right]=\sum_{j=-\infty}^{+\infty}\left(\hat{h}(\lambda,c_{j})-\hat{h}(\lambda,d_{j})\right) \qquad (4.24)$$

where $A=\left\{\tilde{a}\le m_{t_{1}}\le M_{t_{1}}\le\tilde{b},B_{T}\in(-\infty,y_{1}]\right\}$. Using (4.24) to evaluate (4.23) completes the proof. □

**Remark:** *The only differences between the valuation of the PTCO-I and the PTPO-I are the payoffs and the limits of integration. This is what motivated the introduction of the function* $\hat{h}(c,d)$.

## 4.4 Partial Tunnel Options: Type II

### 4.4.1 Partial Tunnel Call Option of Type II

The same type of computations can be used to show that the arbitrage price $V_{\mathrm{II}}^{C}$ for the partial tunnel call option of type II (PTCO-II) can be expressed as

$$\begin{aligned} V_{\mathrm{II}}^{C} = e^{-rT-\frac{1}{2}\mu^2 T}\Bigg( & S\cdot E^{Q}\left[ e^{(\mu+\sigma)\tilde{W}_T}\cdot 1_{\left\{\tilde{a}\le \tilde{W}_u \le \tilde{b}\ \forall u\in[t_1,T], \tilde{W}_T\in[\tilde{k},\infty)\right\}}\right] \\ & - k\cdot E^{Q}\left[ e^{\mu\tilde{W}_T}\cdot 1_{\left\{\tilde{a}\le \tilde{W}_u \le \tilde{b}\ \forall u\in[t_1,T], \tilde{W}_T\in[\tilde{k},\infty)\right\}}\right]\Bigg). \end{aligned} \tag{4.25}$$

**Theorem 4.5** *The price* $V_{\mathrm{II}}^{C}$ *of the* PTCO-II *is given by*

$$V_{\mathrm{II}}^{C} = \sum_{j=-\infty}^{+\infty} e^{-rT-\frac{1}{2}\mu^2 T}\left[ S\cdot\left( w(\mu+\sigma,c_j) - w(\mu+\sigma,d_j)\right) - k\cdot\left( w(\mu,c_j) - w(\mu,d_j)\right)\right] \tag{4.26}$$

where

$$\begin{aligned} w(c,d) = \exp\left(\frac{c^2T}{2} - cd\right)\Bigg( & \left[ N_\rho\left(\frac{\tilde{b}+d-\alpha}{\sqrt{t_1}}, \frac{\tilde{b}-\beta}{\sqrt{T}}\right) - N_\rho\left(\frac{\tilde{a}+d-\alpha}{\sqrt{t_1}}, \frac{\tilde{b}-\beta}{\sqrt{T}}\right)\right] \\ & - \left[ N_\rho\left(\frac{\tilde{b}+d-\alpha}{\sqrt{t_1}}, \frac{\delta-\beta}{\sqrt{T}}\right) - N_\rho\left(\frac{\tilde{a}+d-\alpha}{\sqrt{t_1}}, \frac{\delta-\beta}{\sqrt{T}}\right)\right]\Bigg), \end{aligned} \tag{4.27}$$

$\delta = \max(\tilde{a},\tilde{k})$, $c_j = 2(\tilde{b}-\tilde{a})$, $d_j = 2(\tilde{b}-\tilde{a}) - 2\tilde{b}$, $\alpha = ct_1 - d$ and $\beta = cT - d$.

**Proof:** We have

$$V_{\mathrm{II}}^{C} = e^{-rT-\frac{1}{2}\mu^2 T}\left( S \cdot E^Q\left[ e^{(\mu+\sigma)\tilde{W}_T} \cdot 1_{\left\{\tilde{a}\le \tilde{W}_u \le \tilde{b}\ \forall u\in[t_1,T],\tilde{W}_T\in[\tilde{k},\infty)\right\}}\right] - k \cdot E^Q\left[ e^{\mu\tilde{W}_T} \cdot 1_{\left\{\tilde{a}\le \tilde{W}_u \le \tilde{b}\ \forall u\in[t_1,T],\tilde{W}_T\in[\tilde{k},\infty)\right\}}\right]\right). \tag{4.28}$$

We do not know whether $\tilde{k} \le \tilde{a}$ or $\tilde{a} \le \tilde{k}$. This is the reason $\delta = \max(\tilde{a},\tilde{k})$ is introduced. We may then rewrite (4.28) with $\delta$ yielding

$$V_{\mathrm{II}}^{C} = e^{-rT-\frac{1}{2}\mu^2 T}\left( S \cdot E^Q\left[ e^{(\mu+\sigma)\tilde{W}_T} \cdot 1_{\left\{\tilde{a}\le \tilde{W}_u \le \tilde{b}\ \forall u\in[t_1,T],\tilde{W}_T\in[\delta,\infty)\right\}}\right] - k \cdot E^Q\left[ e^{\mu\tilde{W}_T} \cdot 1_{\left\{\tilde{a}\le \tilde{W}_u \le \tilde{b}\ \forall u\in[t_1,T],\tilde{W}_T\in[\delta,\infty)\right\}}\right]\right). \tag{4.29}$$

We then apply theorem 3.6 to evaluate the expectations which yields (4.26). This completes the proof. □

4.4.2 Partial Tunnel Put Option of Type II

The same type of computations can be used to show that the arbitrage price $V_{II}^{P}$ for the partial tunnel put option of type II (PTPO-II) can be expressed as

$$V_{\mathrm{II}}^{P} = e^{-rT-\frac{1}{2}\mu^2 T}\left( k \cdot E^Q\left[ e^{\mu\tilde{W}_T} \cdot 1_{\left\{\tilde{a}\le \tilde{W}_u \le \tilde{b}\ \forall u\in[t_1,T],\tilde{W}_T\in(\tilde{a},\tilde{k}]\right\}}\right] - S \cdot E^Q\left[ e^{(\mu+\sigma)\tilde{W}_T} \cdot 1_{\left\{\tilde{a}\le \tilde{W}_u \le \tilde{b}\ \forall u\in[t_1,T],\tilde{W}_T\in(\tilde{a},\tilde{k}]\right\}}\right]\right) \tag{4.30}$$

**Theorem 4.6** *The price* $V_{\mathrm{II}}^{P}$ *of the* PTPO-II *is given by*

$$V_{\mathrm{II}}^{P} = \sum_{j=-\infty}^{+\infty} e^{-rT-\frac{1}{2}\mu^2 T}\left[ k\cdot\left(w(\mu,c_j)-w(\mu,d_j)\right) - S\cdot\left(w(\mu+\sigma,c_j)-w(\mu+\sigma,d_j)\right)\right] \tag{4.31}$$

where $w(c,d)$ is as defined as

$$w(c,d)=\exp\left(\frac{c^2T}{2}-cd\right)\left(\left[N_\rho\left(\frac{\tilde{b}+d-\alpha}{\sqrt{t_1}},\frac{\psi-\beta}{\sqrt{T}}\right)-N_\rho\left(\frac{\tilde{a}+d-\alpha}{\sqrt{t_1}},\frac{\psi-\beta}{\sqrt{T}}\right)\right]\right.$$
$$\left.-\left[N_\rho\left(\frac{\tilde{b}+d-\alpha}{\sqrt{t_1}},\frac{\tilde{a}-\beta}{\sqrt{T}}\right)-N_\rho\left(\frac{\tilde{a}+d-\alpha}{\sqrt{t_1}},\frac{\tilde{a}-\beta}{\sqrt{T}}\right)\right]\right),\qquad (4.32)$$

$\psi=\min(\tilde{b},\tilde{k})$, $c_j=2(\tilde{b}-\tilde{a})$, $d_j=2(\tilde{b}-\tilde{a})-2\tilde{b}$, $\alpha=ct_1-d$ and $\beta=cT-d$.

**Proof:** We have

$$V_{\text{II}}^P=e^{-rT-\frac{1}{2}\mu^2T}\left(k\cdot E^Q\left[e^{\mu\tilde{W}_T}\cdot 1_{\left\{\tilde{a}\le\tilde{W}_u\le\tilde{b}\ \forall u\in[t_1,T],\tilde{W}_T\in(\tilde{a},\tilde{k}]\right\}}\right]\right.$$
$$\left.-S\cdot E^Q\left[e^{(\mu+\sigma)\tilde{W}_T}\cdot 1_{\left\{\tilde{a}\le\tilde{W}_u\le\tilde{b}\ \forall u\in[t_1,T],\tilde{W}_T\in(\tilde{a},\tilde{k}]\right\}}\right]\right)\qquad (4.33)$$

Again we do not know whether $\tilde{k}\le\tilde{b}$ or $\tilde{b}\le\tilde{k}$. So we introduce $\psi=\min(\tilde{b},\tilde{k})$ and rewrite (4.33) as

$$V_{\text{II}}^P=e^{-rT-\frac{1}{2}\mu^2T}\left(k\cdot E^Q\left[e^{\mu\tilde{W}_T}\cdot 1_{\left\{\tilde{a}\le\tilde{W}_u\le\tilde{b}\ \forall u\in[t_1,T],\tilde{W}_T\in(\tilde{a},\psi]\right\}}\right]\right.$$
$$\left.-S\cdot E^Q\left[e^{(\mu+\sigma)\tilde{W}_T}\cdot 1_{\left\{\tilde{a}\le\tilde{W}_u\le\tilde{b}\ \forall u\in[t_1,T],\tilde{W}_T\in(\tilde{a},\psi]\right\}}\right]\right).\qquad (4.34)$$

We then apply theorem 3.6 to evaluate the expectations which yields (4.31). This completes the proof. □

# CHAPTER 5
# ANALYSIS OF THE PARTIAL TUNNEL OPTIONS PRICING FORMULAS

The first part of this chapter shows how the partial tunnel options pricing formulas generalize those of the existing options pricing formulas. Two methods are used, both of which are straightforward limiting procedures. In particular, we take the limit of a partial tunnel pricing formula as the upper and lower barriers go to infinity and zero, respectively. We also fix the upper and lower barriers and take the limit as the monitoring time goes to zero. In the case of Type I partial tunnel options, both of these limits turn out to be the pricing formulas for the standard options. However, a subtlety arises in the latter limit for partial tunnel options of Type II. We first revert back to the original expressions for the options prices, as expectations of payoffs. This will greatly simplify the task as we take limits. Second, we take the limit of the new pricing formula given in chapter 4 as an infinite series. Although this requires much more detail and cumbersome analysis, we will see the Black-Scholes formula emerge as the limit of the PTCO-I price as the monitoring period $t_1$ approaches zero. The second part of the chapter provides numerical results and explains the patterns found in the results.

## 5.1 Partial Tunnel Options Pricing Formulas as Extensions of Existing Formulas

### 5.1.1 Limits of Partial Tunnel Options Prices as Expectations

Recall that the arbitrage price $V_1^C$ of a PTCO-I is given by

$$V_1^C = E^P\left[e^{-rT}\cdot 1_{\{a\le S_u\le b\ \forall u\in[0,t_1]\}}(S_T-k)^+\right]. \tag{5.1}$$

Clearly we have

$$\begin{aligned}0 &\le 1_{\{a\le S_u\le b\ \forall u\in[0,t_1]\}}(S_T-k)^+ \\ &\le (S_T-k)^+\end{aligned}. \tag{5.2}$$

Moreover,

$$\lim_{b\to+\infty} 1_{\{a\le S_u\le b\ \forall u\in[0,t_1]\}}(S_T-k)^+ = 1_{\{a\le S_u\ \forall u\in[0,t_1]\}}(S_T-k)^+ \tag{5.3}$$

and

$$\lim_{a\to 0} 1_{\{a\le S_u\ \forall u\in[0,t_1]\}}(S_T-k)^+ = (S_T-k)^+. \tag{5.4}$$

Since these limits are increasing and

$$1_{\{a\le S_u\le b\ \forall u\in[0,t_1]\}}(S_T-k)^+ \ge 0 \tag{5.5}$$

we may use the Monotone Convergence Theorem (twice) to obtain

$$\lim_{\substack{a\to 0\\ b\to+\infty}} E^P\left[1_{\{a\le S_u\le b\ \forall u\in[0,t_1]\}}(S_T-k)^+\right] = E^P\left[(S_T-k)^+\right]. \tag{5.6}$$

That is, the limit of the price of a PTCO-I option as the barriers go off to $\infty$ and $0$ is the price of a standard call option.

**Remark:** *The first application of the Monotone Convergence Theorem shows that the partial tunnel option price converges to that of a partial barrier down and out call as the*

*upper barrier goes to* $\infty$. *Since the limits can be taken in either order, we also see that the partial tunnel option price converges to that of a partial barrier up and out call as the lower barrier goes to* $0$.

Another way to see (5.6) is to take the limit as the length of the monitoring period, $t_1$, goes to zero. In particular, we have

$$\lim_{t_1 \to 0} 1_{\{a \le S_u \le b \ \forall u \in [0,t_1]\}} (S_T - k)^+ = (S_T - k)^+ . \tag{5.7}$$

Since this is also an increasing limit and we have (5.5), we may again apply the Monotone Convergence Theorem to obtain (5.6).

**Remark:** *Clearly we could repeat the same analysis for the PTPO-I. However, in the case of the Type II partial tunnel options, taking the limit as* $t_1$ *approaches* $T$ *does not converge to the price of a standard option. In this case the payoffs are always bounded. This is due to the fact that all of the terms in the limit are uniformly bounded. No matter how close* $t_1$ *is to* $T$, *the barriers still exist at time* $T$ *and therefore impose a bound on the payoffs for call and put options. They do however converge to the prices for call-like and put-like options whose payoffs are given by* $X = 1_{\{a \le S_T \le b\}} (S_T - k)^+$ *and*

$X = 1_{\{a \le S_T \le b\}} (k - S_T)^+$.

5.1.2 Limits of Partial Tunnel Options Formulas

In this section we will focus on the PTCO-I. In particular, we will take the limit of the pricing formula obtained in chapter 4 as the length of the monitoring period $t_1$ approaches zero. The formula is expressed as a doubly infinite series. We will see that all of the terms except for the central term, the term corresponding to $j = 0$, will approach

zero and therefore drop out from the formula. The resulting limit is the classic Black-Scholes formula for a call option. Note that we use the uniform convergence from chapter 3 to interchange the infinite sum with the limit.

The formula for the PTCO-I found in the appendix is given by

$$
\begin{aligned}
V_1^C = \sum_{j=-\infty}^{+\infty} \Bigg[ S \Bigg( & e^{-(\mu+\sigma)c_j} \Bigg[ N_\rho \left( \frac{(\mu+\sigma)t_1 - c_j - \tilde{a}}{\sqrt{t_1}}, \frac{(\mu+\sigma)T - c_j - \tilde{k}}{\sqrt{T}} \right) \\
& \qquad - N_\rho \left( \frac{(\mu+\sigma)t_1 - c_j - \tilde{b}}{\sqrt{t_1}}, \frac{(\mu+\sigma)T - c_j - \tilde{k}}{\sqrt{T}} \right) \Bigg] \\
& - e^{-(\mu+\sigma)d_j} \Bigg[ N_\rho \left( \frac{(\mu+\sigma)t_1 - d_j - \tilde{a}}{\sqrt{t_1}}, \frac{(\mu+\sigma)T - d_j - \tilde{k}}{\sqrt{T}} \right) \\
& \qquad - N_\rho \left( \frac{(\mu+\sigma)t_1 - d_j - \tilde{b}}{\sqrt{t_1}}, \frac{(\mu+\sigma)T - d_j - \tilde{k}}{\sqrt{T}} \right) \Bigg] \Bigg) \\
& - e^{-rT} k \cdot \Bigg( e^{-\mu c_j} \Bigg[ N_\rho \left( \frac{\mu t_1 - c_j - \tilde{a}}{\sqrt{t_1}}, \frac{\mu T - c_j - \tilde{k}}{\sqrt{T}} \right) \\
& \qquad - N_\rho \left( \frac{\mu t_1 - c_j - \tilde{b}}{\sqrt{t_1}}, \frac{\mu T - c_j - \tilde{k}}{\sqrt{T}} \right) \Bigg] \\
& - e^{-\mu d_j} \Bigg[ N_\rho \left( \frac{\mu t_1 - d_j - \tilde{a}}{\sqrt{t_1}}, \frac{\mu T - d_j - \tilde{k}}{\sqrt{T}} \right) \\
& \qquad - N_\rho \left( \frac{\mu t_1 - d_j - \tilde{b}}{\sqrt{t_1}}, \frac{\mu T - d_j - \tilde{k}}{\sqrt{T}} \right) \Bigg] \Bigg) \Bigg]
\end{aligned}
\qquad (5.8)
$$

where $c_j = 2(\tilde{b} - \tilde{a})$, $d_j = 2(\tilde{b} - \tilde{a}) - 2\tilde{b}$ and $\rho = \sqrt{\frac{t_1}{T}}$. Recall that $\tilde{a} = \frac{\ln \frac{a}{S}}{\sigma}$ and $\tilde{b} = \frac{\ln \frac{b}{S}}{\sigma}$. Since $a < S$ and $b > S$ we have $\tilde{a} < 0$ and $\tilde{b} > 0$. First we consider the following quantities

$$
\begin{aligned}
&-c_j - \tilde{a}, \\
&-c_j - \tilde{b}, \\
&-d_j - \tilde{a} \\
&-d_j - \tilde{b}
\end{aligned}
\tag{5.9}
$$

for $j \neq 0$. In particular, we will show that each is either positive or negative according to whether $j$ is positive or negative. We substitute $c_j = 2(\tilde{b} - \tilde{a})$ and $d_j = 2(\tilde{b} - \tilde{a}) - 2\tilde{b}$ into (5.9) so that we have

$$
\begin{aligned}
-c_j - \tilde{a} &= -2j(\tilde{b} - \tilde{a}) - \tilde{a}, \\
-c_j - \tilde{b} &= -2j(\tilde{b} - \tilde{a}) - \tilde{b}, \\
-d_j - \tilde{a} &= -2j(\tilde{b} - \tilde{a}) + 2\tilde{b} - \tilde{a} \\
-d_j - \tilde{b} &= -2j(\tilde{b} - \tilde{a}) + 2\tilde{b} - \tilde{b}.
\end{aligned}
\tag{5.10}
$$

Simplifying yields

$$
\begin{aligned}
-c_j - \tilde{a} &= (2j-1)\tilde{a} - 2j\tilde{b}, \\
-c_j - \tilde{b} &= 2j\tilde{a} - (2j+1)\tilde{b}, \\
-d_j - \tilde{a} &= (2j-1)\tilde{a} - (2j-2)\tilde{b} \\
-d_j - \tilde{b} &= 2j\tilde{a} - (2j-2)\tilde{b}.
\end{aligned}
\tag{5.11}
$$

Observe that if $j > 0$ then all of the quantities in (5.11) are negative. On the other hand, if $j < 0$ then all of the quantities are positive. These observations should be evident from the fact that $\tilde{a} < 0$ and $\tilde{b} > 0$. Now consider the difference

$$
\begin{aligned}
&N_\rho\left(\frac{(\mu+\sigma)t_1 - c_j - \tilde{a}}{\sqrt{t_1}}, \frac{(\mu+\sigma)T - c_j - \tilde{k}}{\sqrt{T}}\right) \\
&\qquad -N_\rho\left(\frac{(\mu+\sigma)t_1 - c_j - \tilde{b}}{\sqrt{t_1}}, \frac{(\mu+\sigma)T - c_j - \tilde{k}}{\sqrt{T}}\right)
\end{aligned}
.
\tag{5.12}
$$

For $j > 0$, the first argument in each bivariate normal distribution function is negative for sufficiently small values of $t_1$. As $t_1$ tends to zero each of these arguments goes to $-\infty$. Thus each distribution function, and hence the entire expression, will go to zero. For $j < 0$, the first arguments are both positive and therefore tend to $+\infty$. Each bivariate distribution function then converges to a (univariate) normal distribution. Since the second arguments in each bivariate distribution function are the same, the difference approaches zero. The same argument shows that each of the three other difference expressions go to zero as well. This leaves us with the term corresponding to $j = 0$. This term simplifies to

$$
\begin{aligned}
&S\Bigg(\Bigg[N_\rho\left(\frac{(\mu+\sigma)t_1-\tilde{a}}{\sqrt{t_1}},\frac{(\mu+\sigma)T-\tilde{k}}{\sqrt{T}}\right)\\
&\qquad -N_\rho\left(\frac{(\mu+\sigma)t_1-\tilde{b}}{\sqrt{t_1}},\frac{(\mu+\sigma)T-\tilde{k}}{\sqrt{T}}\right)\Bigg]\\
&\qquad -e^{-(\mu+\sigma)(-2\tilde{b})}\Bigg[N_\rho\left(\frac{(\mu+\sigma)t_1+2\tilde{b}-\tilde{a}}{\sqrt{t_1}},\frac{(\mu+\sigma)T+2\tilde{b}-\tilde{k}}{\sqrt{T}}\right)\\
&\qquad\qquad -N_\rho\left(\frac{(\mu+\sigma)t_1+2\tilde{b}-\tilde{b}}{\sqrt{t_1}},\frac{(\mu+\sigma)T+2\tilde{b}-\tilde{k}}{\sqrt{T}}\right)\Bigg]\Bigg)\\
&-e^{-rT}k\cdot\Bigg(\Bigg[N_\rho\left(\frac{\mu t_1-\tilde{a}}{\sqrt{t_1}},\frac{\mu T-\tilde{k}}{\sqrt{T}}\right)\\
&\qquad -N_\rho\left(\frac{\mu t_1-\tilde{b}}{\sqrt{t_1}},\frac{\mu T-\tilde{k}}{\sqrt{T}}\right)\Bigg]\\
&\qquad -e^{-\mu(-2\tilde{b})}\Bigg[N_\rho\left(\frac{\mu t_1+2\tilde{b}-\tilde{a}}{\sqrt{t_1}},\frac{\mu T+2\tilde{b}-\tilde{k}}{\sqrt{T}}\right)\\
&\qquad\qquad -N_\rho\left(\frac{\mu t_1+2\tilde{b}-\tilde{b}}{\sqrt{t_1}},\frac{\mu T+2\tilde{b}-\tilde{k}}{\sqrt{T}}\right)\Bigg]
\end{aligned}
\quad . \qquad (5.13)
$$

Eight bivariate normal distribution functions appear in this term. We use the same analysis as we did above. Recall that $\tilde{a} < 0$ and $\tilde{b} > 0$. Note also that $2\tilde{b} - \tilde{a} > 0$. As $t_1$ goes to zero, the first and fifth bivariate distribution functions converge to a (univariate) normal. The second and sixth each go to zero. The four remaining distribution functions each go to 1 and therefore cancel each other out. Thus the first term is simply

$$S \cdot N\left( \frac{(\mu + \sigma)T - \tilde{k}}{\sqrt{T}} \right) - e^{-rT} k \cdot N\left( \frac{\mu T - \tilde{k}}{\sqrt{T}} \right). \tag{5.14}$$

Replacing $\mu = \dfrac{1}{\sigma}\left( r - \dfrac{\sigma^2}{2} \right)$ and $\tilde{k} = \dfrac{\ln\left(\frac{k}{S}\right)}{\sigma}$ we may simplify (5.14) to recover the classic Black-Scholes formula.

## 5.2 Numerical Results

In this section we examine some features of the PTCO-I and PTCO-II. Naturally, the PTPO-I and PTPO-II will share many of these features, so they will be left out. For each of the partial tunnel call options, we will numerically illustrate the analytic results of section 5.1. We will also examine the effects that the volatility and the length of the monitoring period have on the price of each option.

### 5.2.1 The Partial Tunnel Call Option: Type I

Consider a PTCO-I where we have the following parameters: initial stock price $S = 55$, volatility $\sigma = .2$, expiry $T = 1$, monitoring time $t_1 = .5$, strike price $k = 65$, lower barrier $a = 40$, upper barrier $b = 80$ and interest rate $r = .06$. We will first show that as the upper and lower barriers of this option approach infinity and zero, respectively,

the price of the option will approach that of the standard call option with initial stock price $S = 55$, volatility $\sigma = .2$, expiry $T = 1$, strike price $k = 65$ interest rate $r = .06$. The appropriate price for this standard call option is \$2.166. We price the same PTCO-I with upper barriers ranging from 80 to 120 and lower barriers ranging from 40 to 5. The results are presented in figure 5.1. Observe that the price is the same as the Black-Scholes price when the lower and upper barriers have been moved to 40 and 120, respectively.

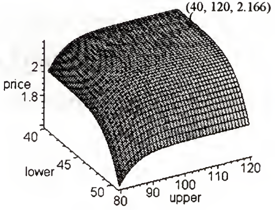


Figure 5.1: Varying upper and lower barriers for a PTCO-I

We now consider the case above but let the length of the monitoring time $t_1$ go to zero. In order to illustrate the effect that this has on the option price, we will use the same parameters as above except that the upper and lower barriers will be 70 and 50, respectively. We observe the prices with $t_1$ ranging from 0 to 1. The results are displayed in figure 5.2. Observe once again that the price approaches the standard Black-Scholes price as $t_1$ approaches zero.

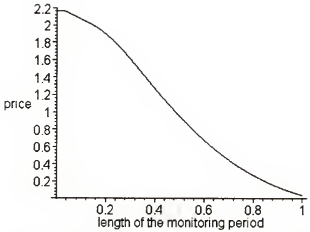


Figure 5.2: Varying the length of the monitoring period for a PTCO-I

We now observe the affect that the volatility $\sigma$ and the length of the monitoring period $t_1$ have on the price. In the case of the standard call option, increasing the volatility increases the price of the option. However, in the case of the partial tunnel options, increasing the volatility only increases the price up to a point. After this point, the probability of hitting a barrier and having a zero payoff is so high that increasing the volatility only decreases the chance of not hitting a barrier and having a positive payoff. Thus increasing the volatility after this point only serves to decrease the price. This phenomenon is present in both Type I and Type II partial tunnel options.

The length of the monitoring period is also a factor in the price. For partial tunnel options of Type I the monitoring period begins at time zero and ends at some time $t_1$. Thus the length of the monitoring period is simply $t_1$. Since increasing $t_1$ only increases the chance of getting a zero payoff, we see that the price will decrease as we increase $t_1$.

Here we take the case where the intial stock price is $S = 20$, $\sigma = .18$, r = .06, $T = 2$, the upper barrier is 45, the lower barrier is 15 and the strike price is $k = 25$. Figure 5.3 plots the values of the prices as the volatility ranges from .05 to .95 and the monitoring time ranges from 0 to 2.

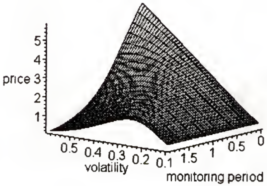


Figure 5.3: Varying $\sigma$ and $t_1$ for a PTCO-I

Notice that as the length of the monitoring period $t_1$ gets closer to zero, the price appears to increase as $\sigma$ increases. This occurs as a result of the PTCO-I converging to a regular call option where the price always increases as volatility increases. This phenomenon will not occur in options of Type II.

5.2.2 The Partial Tunnel Call Option: Type II

Consider a PTCO-II where we have the following parameters: initial stock price $S = 55$, volatility $\sigma = .2$, expiry $T = 1$, monitoring time $t_1 = .5$, strike price $k = 65$, lower barrier $a = 40$, upper barrier $b = 80$ and interest rate $r = .06$. We show that as the upper and lower barriers of this option approach infinity and zero, respectively, the price

of the option will approach that of the standard call option with the same parameters. We will not be taking the limit as the length of the monitoring time goes to zero for the reasons mentioned in the above remark. Recall that the appropriate price for this standard call option is $\$2.166$. We price the same PTCO-II with upper barriers ranging from 80 to 125 and lower barriers ranging from 55 to 25. The results are presented in figure 5.4. Observe that the price is the same (to at least three digits of accuracy) as the Black-Scholes price when the lower and upper barriers have been moved to 25 and 125, respectively.

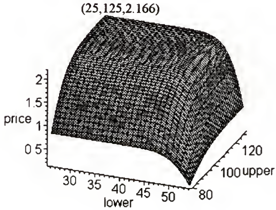


Figure 5.4: Varying upper and lower barriers for a PTCO-II

We again observe the effect that the volatility $\sigma$ and the length of the monitoring period have on the price options. Now for options of Type II, the monitoring period begins at $t_1$ and ends at time $T$. Therefore the length of the monitoring period in this case is $T - t_1$. As mentioned above, increasing the volatility only increases the price up to a point and then the price begins to decrease. We now look at the the case where the intial stock price is $S = 55$, $\sigma = .2$, r = .06, $T = 2$, the upper barrier is 100, the lower barrier is

30 and the strike price is $k = 65$. The volatility ranges from .05 to .30 and the length of the monitoring period ranges from 0 to 2. These results are presented in figure 5.5 below. Note that the axis for the monitoring period represents the length of the monitoring period $T - t_1$ which was $t_1$ in the case of the Type I option above.

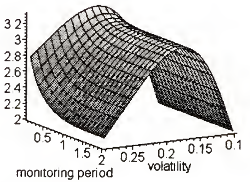


Figure 5.5: Varying $\sigma$ and $T - t_1$ for a PTCO-II

Observe the plot where the length of the monitoring period $T - t_1$ gets closer to zero. As $\sigma$ increases, the price still only increases to a point and then decreases. This phenomenon is different from that of the PTCO-I where the price appeared to strictly increase as $\sigma$ increased for small values of $t_1$.

## CHAPTER 6
## THE GET-OUT OPTION

### 6.1 Description

The get-out option is based on a strategy that will get one out of the market when a certain trigger is initiated. Consider a bullish investor who, for some reason, cannot monitor the market and wishes to implement a strategy that would exit him from further exposure if one of his holdings were to go below a certain level. The get-out option depends upon two underlying securities and its payoff is determined by whether or not these securities go below a certain level during the lifetime of the option. There are three possible occurrences:

1. Stock 1 hits a lower barrier $a$ and triggers the payoff of an exercised call on Stock 2 before time T.
2. Stock 2 hits a lower barrier $b$ and triggers the payoff of an exercised call on Stock 1 before time T.
3. Neither stock hits its respective barrier before the expiry time $T$ and the option returns the payoffs of call options on both stocks at time T.

This option is different from the options of previous chapters in that it depends on two underlying stocks. As a result, the Black-Scholes setup from chapter 1 will no longer suffice. We introduce a generalization of this model that will enable us to price this new option appropriately.

### 6.2 The Multidimensional Black-Scholes Setup

#### 6.2.1 The Model

We must now use the multidimensional (2 dimensional) Black-Scholes Model [9]. In this model we have the filtered probability space $(\Omega, \mathcal{F}, P)$ in which the processes

$$\begin{aligned} dS_t &= S_t\left(u_t \cdot dt + \Sigma_t \cdot dW_t\right) \\ dB_t &= r_t B_t dt \end{aligned} \tag{6.1}$$

are defined. Here $S_t$ is a (2x1) vector of stock price processes, $u_t$ is a (2x1) vector of drifts, $\Sigma_t$ is the volatility (2x2) matrix and $W_t$ is a multidimensional (2x1) Brownian motion under the probability measure $P$. We also have the natural filtration $\mathcal{F}_t$, that is the filtration generated by the 2 dimensional process $S_t$.

**Definition 6.1** A process $\gamma$ for which $r_t \cdot 1 - u_t = \Sigma_t \cdot \gamma_t$ is called the *market price for risk*.

The existence of this process in conjunction with Girsanov's Theorem gives rise to a martingale measure for our model. Recall that the martingale measure is simply the measure under which the discounted stock price process is a martingale. Since the stock price process is 2 dimensional, this implies that each component of the process $e^{-r't}S_t$ is a martingale. For simplicity we consider the case where $u_t = u$, $\Sigma_t = \Sigma$ is nonsingular and $r_t = r$. In particular, the drift, volatility and interest rate do not depend on time. Under these assumptions it is clear that the process $\gamma$ indeed exists and is given by $\gamma = \Sigma^{-1}(r \cdot 1 - u)$. This model together with these assumptions is referred to as the *classic* Black-Scholes model.

6.2.2 The Martingale Measure

As before, we begin with the martingale measure $P$. Under this measure we have

$$d\left(\frac{S_t^i}{B_t}\right) = \left(\frac{S_t^i}{B_t}\right)\Sigma^i \cdot dW_t \tag{6.2}$$

for i=1,2 where $\Sigma^i$ denotes the i[th] row of the matrix $\Sigma$. In particular, the discounted stock price process is a martingale under $P$. The solution to this stochastic differential equation is given by

$$S_t^i = S^i \exp\left(\Sigma^i \cdot W_t + \left(r - \frac{\sigma_{i1}{}^2 + \sigma_{i2}{}^2}{2}\right)t\right) \tag{6.3}$$

when $\Sigma$ is given by

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}. \tag{6.4}$$

We may write out (6.3) as

$$S_t^i = S^i \exp\left(\sigma_{i1} W_t^1 + \sigma_{i2} W_t^2 + \left(r - \frac{\sigma_{i1}{}^2 + \sigma_{i2}{}^2}{2}\right)t\right). \tag{6.5}$$

6.2.3 The Payoff

The discounted payoff $X$ for the get-out option can be expressed as the sum of three exclusive payoffs that correspond to the three possible outcomes set forth in 5.1. Set $\tau_1 = \inf\left\{s > 0 : S_s^1 = a\right\}$, $\tau_2 = \inf\left\{s > 0 : S_s^2 = b\right\}$ and define the following sets accordingly:

$$\begin{aligned} A_1 &= \{\tau_1 < \tau_2;\ \ \tau_1 < T\} \\ A_2 &= \{\tau_2 < \tau_1;\ \ \tau_2 < T\} \\ A_3 &= \{\tau_1 \geq T;\ \ \tau_2 \geq T\} \end{aligned} \tag{6.6}$$

Clearly we have $P(A_1 \cup A_2 \cup A_3) = 1$ and $P(A_i \cap A_j) = 0$ for $i \neq j$. So the discounted payoff $X$ may be expressed as

$$X = 1_{A_1}\left(e^{-r\tau_1}\cdot\left(S^2_{\tau_1}-k_2\right)^+\right) + 1_{A_2}\left(e^{-r\tau_2}\cdot\left(S^1_{\tau_2}-k_1\right)^+\right) + 1_{A_3}\left(e^{-rT}\left(\left(S^1_T-k_1\right)^+ + \left(S^2_T-k_2\right)^+\right)\right) \quad . \tag{6.7}$$

Our goal now is to price this option. Just as before, we need only take the expectation of $X$ under the martingale measure $P$.

## 6.3 Pricing

### 6.3.1 The Independent Case

This is the simpler of the two cases. For this case we assume that the two stock price processes are independent. Consequently, the non-diagonal entries in the volatility matrix $\Sigma$ must be zero. Otherwise the processes would both contain the same nonzero Brownian motion terms and therefore be correlated. If we denote the diagonal entries of the volatility matrix by $\sigma_1 = \sigma_{11}$ and $\sigma_2 = \sigma_{22}$ then we may write the stock price processes as

$$S^i_t = S^i \exp\left(\sigma_i W^i_t + \mu_i t\right) \tag{6.8}$$

where $\mu_i = r - \frac{\sigma_i^2}{2}$ for i=1,2. Recall from (6.6) and (6.7) that we will be taking the expectation of

$$X = 1_{A_1}\left(e^{-r\tau_1}\cdot\left(S^2_{\tau_1}-k_2\right)^+\right) + 1_{A_2}\left(e^{-r\tau_2}\cdot\left(S^1_{\tau_2}-k_1\right)^+\right) + 1_{A_3}\left(e^{-rT}\left(\left(S^1_T-k_1\right)^+ + \left(S^2_T-k_2\right)^+\right)\right) \quad . \tag{6.9}$$

where

$$\begin{aligned} A_1 &= \{\tau_1 < \tau_2;\ \tau_1 < T\} \\ A_2 &= \{\tau_2 < \tau_1;\ \tau_2 < T\}. \\ A_3 &= \{\tau_1 \geq T;\ \tau_2 \geq T\} \end{aligned} \tag{6.10}$$

We begin with the first term. We wish to compute

$$E\left[1_{A_1}\left(e^{-r\tau_1}\cdot\left(S_{\tau_1}^2 - k_2\right)^+\right)\right] \tag{6.11}$$

under the martingale measure $P$. Recall that the process $S_u^2$ is defined as

$$S_u^2 = S^2 \exp\left(\sigma_2 W_u^2 + \mu_2 u\right). \tag{6.12}$$

Define the processes $X_u^i$ by

$$X_u^i = \sigma_i W_u^i + \mu_i u \text{ for i=1,2.} \tag{6.13}$$

Then we have

$$S_u^i = S^i \exp\left(X_u^i\right) \text{ for i=1,2.} \tag{6.14}$$

Now write the set $A_1$ as $A_1 = A_1' \cap A_1''$ where $A_1' = \{\tau_1 < \tau_2\}$ and $A_1'' = \{\tau_1 < T\}$. If we have the density $P(\tau_1 \in du, \tau_1 < \tau_2, X_{\tau_1}^2 \in dy)$ then we will be able to express (6.11) as a Lebesgue integral. First note that $A_1' = \left\{m_{\tau_1}^2 > \ln\left(\frac{b}{S^2}\right)\right\}$ where $m_t^2$ is the running minimum for the process $X_t^2$. Thus we may write

$$P(\tau_1 \in du, \tau_1 < \tau_2, X_{\tau_1}^2 \in dy) = P(\tau_1 \in du, m_{\tau_1}^2 > \tilde{b}, X_{\tau_1}^2 \in dy) \tag{6.15}$$

where $\tilde{b} = \ln\left(\frac{b}{S^2}\right)$. Using methods in [12] and the independence we have

$$\begin{aligned} P(\tau_1 \in du, m_{\tau_1}^2 > \tilde{b}, X_{\tau_1}^2 \in dy) &= P(\tau_1 \in du, m_u^2 > \tilde{b}, X_u^2 \in dy) \\ &= P(\tau_1 \in du)\cdot P(m_u^2 > \tilde{b}, X_u^2 \in dy) \end{aligned} \tag{6.16}$$

The densities on the right hand side are found in [7]. The density $P\left(X_u^2 \in dx, m_u^2 > \tilde{b}\right)$ is given by

$$P\left(X_u^2 \in dx, m_u^2 > \tilde{b}\right) =$$

$$\frac{1}{\sqrt{2\pi\sigma^2 u}}\left[\exp\left(\frac{(x-\mu_2 u)^2}{2\sigma^2 u}\right) - \exp\left(\frac{2\mu_2\tilde{b}}{\sigma^2}\right)\exp\left(\frac{\left(2\tilde{b}-x-\mu_2 u\right)^2}{2\sigma^2 u}\right)\right]dx \quad (6.17)$$

Recall that $\tau_1 = \inf\left\{s > 0 : S_s^1 = a\right\}$. We substitute $S_s^1 = S^1\exp\left(X_s^1\right)$ and rewrite this as

$$\begin{aligned}\tau_1 &= \inf\left\{s > 0 : S_s^1 = a\right\}\\ &= \inf\left\{s > 0 : S^1\exp\left(X_s^1\right) = a\right\}\\ &= \inf\left\{s > 0 : X_s^1 = \tilde{a}\right\}\\ &= \inf\left\{s > 0 : W_s^1 + \frac{\mu_1}{\sigma_1}s = \tilde{a}\right\}\end{aligned} \quad (6.18)$$

where $\tilde{a} = \dfrac{\ln\left(\dfrac{S^1}{a}\right)}{\sigma_1}$. Then we have the density

$$P(\tau_1 \in du) = \frac{|\tilde{a}|}{\sqrt{2\pi u^3}}\exp\left[-\frac{\left(\tilde{a}-\dfrac{\mu_1}{\sigma_1}u\right)^2}{2u}\right]du. \quad (6.19)$$

So we may express (6.11) as

$$\begin{aligned}&E\left[1_{A_1}\left(e^{-r\tau_1}\cdot\left(S_{\tau_1}^2-k_2\right)^+\right)\right]\\ &\quad= \int_0^T\int_{-\infty}^{+\infty} e^{-ru}\left(S^2e^y-k_2\right)^+ P(\tau_1 \in du, \tau_1 < \tau_2, X_{\tau_1}^2 \in dy)\\ &\quad= \int_0^T\int_{-\infty}^{+\infty} e^{-ru}\left(S^2e^y-k_2\right)^+ P(\tau_1 \in du)P\left(X_u^2 \in dy, m_u^2 > \tilde{b}\right)\end{aligned} \quad (6.20)$$

We remove the $(\cdot)^+$ notation by changing the lower limit of integration of $y$ to

$\tilde{k}_2 = \ln\left(\frac{k_2}{S^2}\right)$ so that the quantity inside of $(\cdot)^+$ above is strictly positive. Then (6.20)

becomes

$$\int_0^T \int_{\tilde{k}_2}^{+\infty} e^{-ru} \left(S^2 e^y - k_2\right)^+ P(\tau_1 \in du) P\left(X_u^2 \in dy, m_u^2 > \tilde{b}\right). \tag{6.21}$$

Notice that the second expectation is the same as the first with the roles of stock 1 and stock 2 reversed. Thus we can write it as

$$\int_0^T \int_{\tilde{k}_1}^{+\infty} e^{-ru} \left(S^1 e^y - k_1\right)^+ P(\tau_2 \in du) P\left(X_u^1 \in dy, m_u^1 > \tilde{a}\right) \tag{6.22}$$

where $\tilde{k}_1 = \ln\left(\frac{k_1}{S^1}\right)$ and $\tilde{a} = \ln\left(\frac{a}{S^1}\right)$.

Lastly, we need to compute the value of the third expectation

$$\begin{aligned} &E\left[1_{A_3}\left(e^{-rT}\left(\left(S_T^1 - k_1\right)^+ + \left(S_T^2 - k_2\right)^+\right)\right)\right] \\ &\qquad = e^{-rT} \cdot E\left[1_{A_3}\left(\left(S_T^1 - k_1\right)^+ + \left(S_T^2 - k_2\right)^+\right)\right] \end{aligned} \tag{6.23}$$

We now distribute the indicator function to each of the terms in the parentheses to compute each expectation. The first term is

$$e^{-rT} \cdot E\left[1_{A_3}\left(S_T^1 - k_1\right)^+\right]. \tag{6.24}$$

Recall that $A_3 = \{\tau_1 \geq T;\ \tau_2 \geq T\}$. We may write $A_3$ as $A_3 = A_3' \cap A_3''$ where $A_3' = \{\tau_1 > T\}$ and $A_3'' = \{\tau_2 > T\}$. We break up the indicator function accordingly and express (6.24) as

$$e^{-rT} \cdot E\left[1_{A_3'} \cdot 1_{A_3''}\left(S_T^1 - k_1\right)^+\right]. \tag{6.25}$$

We may factor this into two separate expectations using the independence so that (6.25) becomes

$$e^{-rT} \cdot E\left[1_{A_3''}\right] \cdot E\left[1_{A_3'}\left(S_T^1 - k_1\right)^+\right]. \tag{6.26}$$

Note that we have

$$\begin{aligned} E\left[1_{A_3''}\right] &= P(A_3'') \\ &= P\left(\tau_2 > T\right) \\ &= \frac{\left|\tilde{b}\right|}{\sqrt{2\pi t^3}} \int_T^{+\infty} \exp\left( -\frac{\left(\tilde{b} - \frac{\mu_2}{\sigma_2} s\right)^2}{2s} \right) ds \end{aligned} \tag{6.27}$$

The last equality is given in [7]. Thus (6.26) is simply the price of a down and out barrier option multiplied by the probability in (6.27). This formula is given in [9]. The second term in the expectation (6.23) is the same with the roles reversed.

**Remark:** *Notice that the above analysis did not use the change of measure employed in earlier chapters to remove the drift. The reason for this was that the densities used already accounted for the processes with drift.*

6.3.2 The Dependent Case

Just as before, under the martingale measure $P$ the stock price processes are

$$S_t^i = S^i \exp\left( \sigma_{i1} W_t^1 + \sigma_{i2} W_t^2 + \left( r - \frac{\sigma_{i1}^{\ 2} + \sigma_{i2}^{\ 2}}{2} \right) t \right) \text{ for i=1,2} \tag{6.28}$$

where $W_t^1$ and $W_t^2$ are standard Brownian motions . Moreover, we are still trying to compute the expectation (under $P$) of the discounted payoff $X$ where

$$
\begin{aligned}
X = {} & 1_{A_1}\left(e^{-r\tau_1}\cdot\left(S_{\tau_1}^2 - k_2\right)^+\right) \\
& +1_{A_2}\left(e^{-r\tau_2}\cdot\left(S_{\tau_2}^1 - k_1\right)^+\right) \\
& +1_{A_3}\left(e^{-rT}\left(\left(S_T^1 - k_1\right)^+ + \left(S_T^2 - k_2\right)^+\right)\right)
\end{aligned}
\tag{6.29}
$$

and

$$
\begin{aligned}
A_1 &= \{\tau_1 < \tau_2;\ \tau_1 < T\} \\
A_2 &= \{\tau_2 < \tau_1;\ \tau_2 < T\}. \\
A_3 &= \{\tau_1 \geq T;\ \tau_2 \geq T\}
\end{aligned}
\tag{6.30}
$$

Recall that $\tau_1 = \inf\{s > 0 : S_s^1 = a\}$ and $\tau_2 = \inf\{s > 0 : S_s^2 = b\}$. This time we will remove the drift with a change of measure. Notice that we must remove the drift term from each of the processes in (6.28). We will change to a measure $Q$ so that

$$
\begin{aligned}
\tilde{W}_t^1 &= W_t^1 + \mu_1 t \\
\tilde{W}_t^2 &= W_t^2 + \mu_2 t
\end{aligned}
\tag{6.31}
$$

where $\tilde{W}_t^1$ and $\tilde{W}_t^2$ are standard Brownian motions under $Q$. However, the drift constants $\mu_1$ and $\mu_2$ must be chosen so that they satisfy

$$
\begin{aligned}
\sigma_{i1}W_t^1 + \sigma_{i2}W_t^2 &= \sigma_{i1}\left(\tilde{W}_t^1 - \mu_1 t\right) + \sigma_{i2}\left(\tilde{W}_t^2 - \mu_2 t\right) \\
&= \sigma_{i1}\tilde{W}_t^1 + \sigma_{i2}\tilde{W}_t^2 - \sigma_{i1}\mu_1 t - \sigma_{i2}\mu_2 t \\
&= \sigma_{i1}\tilde{W}_t^1 + \sigma_{i2}\tilde{W}_t^2 - \left(\sigma_{i1}\mu_1 - \sigma_{i2}\mu_2\right)t \\
&= \sigma_{i1}\tilde{W}_t^1 + \sigma_{i2}\tilde{W}_t^2 - \left(r - \frac{\sigma_{i1}^2 + \sigma_{i2}^2}{2}\right)t
\end{aligned}
\tag{6.32}
$$

for i=1,2. In particular, $\mu_1$ and $\mu_2$ must satisfy the equations

$$
\begin{aligned}
\sigma_{11}\mu_1 + \sigma_{12}\mu_2 &= r - \frac{\sigma_{11}^2 + \sigma_{12}^2}{2} \\
\sigma_{21}\mu_1 + \sigma_{22}\mu_2 &= r - \frac{\sigma_{21}^2 + \sigma_{22}^2}{2}
\end{aligned}
\tag{6.33}
$$

Solving these equations for $\mu_1$ and $\mu_2$ yields

$$\mu_1 = \frac{2r(\sigma_{22}-\sigma_{12})-\sigma_{22}(\sigma_{11}^2+\sigma_{12}^2)+\sigma_{12}(\sigma_{21}^2+\sigma_{22}^2)}{2(\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21})}$$

$$\mu_2 = \frac{2r(\sigma_{11}-\sigma_{21})-\sigma_{11}(\sigma_{21}^2+\sigma_{22}^2)+\sigma_{21}(\sigma_{11}^2+\sigma_{12}^2)}{2(\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21})} \quad (6.34)$$

So we define a new measure $Q$ via the Radon-Nikodym derivative

$$\frac{dP}{dQ} = \exp\left(\mu_1\tilde{W}_T^1 + \mu_2\tilde{W}_T^2 - \frac{\mu_1^2+\mu_2^2}{2}T\right). \quad (6.35)$$

Under the measure $Q$ the stock price processes are

$$S_t^i = S^i \exp\left(\sigma_{i1}\tilde{W}_t^1 + \sigma_{i2}\tilde{W}_t^2\right) \text{ for i=1,2} \quad (6.36)$$

where $\tilde{W}_t^1$ and $\tilde{W}_t^2$ are standard Brownian motions under $Q$. We may now compute the expectation of $X$ from (6.29) as

$$E^P[X] = E^Q\left[\frac{dP}{dQ}X\right] \quad (6.37)$$

Recall that $\tau_1 = \inf\{s>0 : S_s^1 = a\}$. We may rewrite this as

$$\begin{aligned}\tau_1 &= \inf\left\{s>0 : S^1\exp\left(\sigma_{11}\tilde{W}_s^1 + \sigma_{12}\tilde{W}_s^2\right) = a\right\} \\ &= \inf\left\{s>0 : \sigma_{11}\tilde{W}_s^1 + \sigma_{12}\tilde{W}_s^2 = \tilde{a}\right\}\end{aligned} \quad (6.38)$$

where $\tilde{a} = \ln\left(\frac{a}{S^1}\right)$. Similarly, we may write $\tau_2 = \inf\left\{s>0 : \sigma_{21}\tilde{W}_s^1 + \sigma_{22}\tilde{W}_s^2 = \tilde{b}\right\}$ where $\tilde{b} = \ln\left(\frac{b}{S^2}\right)$.

We will make two substitutions, a translation and a rotation, so that we may use the joint densities found in [6] to compute the expectation in (6.37). Note first that the stopping times $\tau_1$ and $\tau_2$ correspond geometrically to the first time a two dimensional Brownian motion $\tilde{W}_t = \begin{pmatrix} \tilde{W}_t^1 \\ \tilde{W}_t^2 \end{pmatrix}$, starting at the origin, first hits either one of the lines

$$\begin{array}{ll} L_1: & \sigma_{11}x + \sigma_{12}y = \tilde{a} \\ L_2: & \sigma_{21}x + \sigma_{22}y = \tilde{b} \end{array}. \tag{6.39}$$

Note that $\tilde{a} < 0$ and $\tilde{b} < 0$ since $a < S^1$ and $b < S^2$. So the intersection of these lines lies in the third quadrant. The point of intersection $\bar{p} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ of these two lines is given by

$$\begin{aligned} x_0 &= \frac{\sigma_{22}\tilde{a} - \sigma_{12}\tilde{b}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}} \\ y_0 &= \frac{\sigma_{11}\tilde{b} - \sigma_{21}\tilde{a}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}} \end{aligned}. \tag{6.40}$$

Denote the angle between the two lines as $\alpha$ (note that this is the obtuse angle containing the origin since both lines have negative slopes).

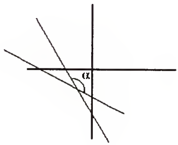


Figure 6.1: Original two lines in the plane

The densities in [6] are for a process that starts at specified point in the plane and then hits one of two lines. One of these lines is the x-axis. The other line goes through the origin and has a negative slope. The point at which the process starts must be located in the wedge that includes the first quadrant. Thus we will need to relocate the intersection of our two lines to the origin and then rotate the plane so that one of the lines coincides with the x-axis. In this manner we will be able to use the densities from [6].

We now make the first substitution

$$X_t = \tilde{W}_t - \bar{p} \tag{6.41}$$

so that $X_t$ is a Brownian motion starting at $-\bar{p}$. This translation moves the intersection of the two lines to the origin. The slopes of line 1 and line 2 are $m_1 = -\frac{\sigma_{11}}{\sigma_{12}}$ and $m_2 = -\frac{\sigma_{21}}{\sigma_{22}}$ respectively. We may suppose without loss of generality that $m_1 > m_2$. Note that we could simply switch the roles of stock 1 and stock 2 to accomplish this. However, the assumption that the slopes are not equal may not be as easy to see. If the slopes are equal then the stocks are perfectly correlated. Since this case is of no interest we may make the assumption that the slopes are indeed not equal.

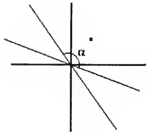


Figure 6.2: Translated plane

The next substitution rotates the plane counter clockwise so that line 1 coincides with the x-axis. Since the slope of line 1 is $m_1 = -\frac{\sigma_{11}}{\sigma_{12}}$, the angle that it makes between itself and the positive x-axis is

$$\theta = \tan^{-1}\left(-\frac{\sigma_{11}}{\sigma_{12}}\right). \tag{6.42}$$

Note that we have $\theta < 0$ since $-\frac{\sigma_{11}}{\sigma_{12}} < 0$. Note also that the angle between line 2 and the x-axis is also negative. Thus the angle is given by

$$\alpha = \pi - \tan^{-1}\left(-\frac{\sigma_{21}}{\sigma_{22}}\right) + \tan^{-1}\left(-\frac{\sigma_{11}}{\sigma_{12}}\right). \tag{6.43}$$

We have $\alpha \geq \frac{\pi}{2}$ since both lines have negative slopes. Define a new process $Y_t$ by

$$Y_t = R_{-\theta} \cdot X_t \tag{6.44}$$

where $R_{-\theta}$ is defined as

$$R_{-\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{6.45}$$

The matrix $R_{-\theta}$ will rotate the plane counterclockwise by an angle of $-\theta > 0$. Thus $Y_t$ is a two dimensional Brownian motion starting at the point $R_{-\theta} \cdot (-\bar{p}) = -R_{-\theta}\bar{p}$. Recall that the original process $\tilde{W}_t$ starts at the origin and the intersection of the two lines is in the third quadrant. Thus the translated and the rotated process $Y_t$ starts in either the first or second quadrant. Moreover, line 1 becomes the x-axis and line 2 creates an obtuse wedge that includes the first quadrant and a portion of the second quadrant. This comes from the fact that the original lines both have non-positive slopes. Note that one of the slopes, but

not both, may be zero. In this case the angle between line 1 and the positive x-axis is zero. Thus the rotation matrix above turns out to be the identity matrix in this case.

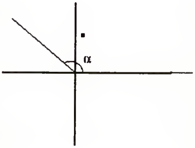


Figure 6.3: Translated and rotated plane

Now the process $Y_t$ may be expressed as

$$\begin{aligned} Y_t &= R_{-\theta} \cdot X_t \\ &= R_{-\theta} \cdot \left(\tilde{W}_t - \bar{p}\right) \\ &= R_{-\theta} \cdot \tilde{W}_t - R_{-\theta} \cdot \bar{p} \end{aligned} \quad . \qquad (6.46)$$

Solving backward for $\tilde{W}_t$ yields

$$\begin{aligned} \tilde{W}_t &= R_{-\theta}^{-1} \left(Y_t + R_{-\theta} \cdot \bar{p}\right) \\ &= R_{-\theta}^{-1} \cdot Y_t + \bar{p} \end{aligned} \quad . \qquad (6.47)$$

We now look at what the individual processes $\tilde{W}_t^1$ and $\tilde{W}_t^2$ look like in terms of the process $Y_t$. So we write out (6.47) as

$$\begin{aligned} \tilde{W}_t &= R_{-\theta}^{-1} \cdot Y_t + \bar{p} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} Y_t^1 \\ Y_t^2 \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta \cdot Y_t^1 - \sin\theta \cdot Y_t^2 + x_0 \\ \sin\theta \cdot Y_t^1 + \cos\theta \cdot Y_t^2 + y_0 \end{pmatrix} \end{aligned} \qquad (6.48)$$

We define the functions $f$ and $g$ by

$$f(x,y) = x \cdot \cos\theta - y \cdot \sin\theta + x_0 \tag{6.49}$$

and

$$g(x,y) = x \cdot \sin\theta + y \cdot \cos\theta + y_0 . \tag{6.50}$$

Then we have

$$\tilde{W}_t = \begin{pmatrix} \tilde{W}_t^1 \\ \tilde{W}_t^2 \end{pmatrix} = \begin{pmatrix} f\left(Y_t^1, Y_t^2\right) \\ g\left(Y_t^1, Y_t^2\right) \end{pmatrix}. \tag{6.51}$$

This will help us write the increasingly large expressions more compactly. Now the stopping time $\tau_1$ defined as

$$\tau_1 = \inf\left\{s > 0 : \sigma_{11}\tilde{W}_s^1 + \sigma_{12}\tilde{W}_s^2 = \tilde{a}\right\} \tag{6.52}$$

may be expressed as

$$\begin{aligned} \tau_1 = \inf\Big\{s > 0 : \sigma_{11}&\left(\cos\theta \cdot Y_s^1 - \sin\theta \cdot Y_s^2 + x_0\right) \\ &+\sigma_{12}\left(\sin\theta \cdot Y_s^1 + \cos\theta \cdot Y_s^2 + y_0\right) = \tilde{a}\Big\} \end{aligned}. \tag{6.53}$$

A simple calculation shows that

$$\sigma_{11}x_0 + \sigma_{12}y_0 = \tilde{a} . \tag{6.54}$$

Thus we may write (6.53) as

$$\begin{aligned} \tau_1 &= \inf\left\{s > 0 : \sigma_{11}\left(\cos\theta \cdot Y_s^1 - \sin\theta \cdot Y_s^2\right) + \sigma_{12}\left(\sin\theta \cdot Y_s^1 + \cos\theta \cdot Y_s^2\right) = 0\right\} \\ &= \inf\left\{s > 0 : \left(\sigma_{11}\cos\theta + \sigma_{12}\sin\theta\right)Y_s^1 + \left(\sigma_{12}\cos\theta - \sigma_{11}\sin\theta\right)Y_s^2 = 0\right\} \end{aligned}. \tag{6.55}$$

Moreover, $\theta = \tan^{-1}\left(-\dfrac{\sigma_{11}}{\sigma_{12}}\right)$, so we may compute the trigonometric functions $\sin\theta$ and $\cos\theta$ as

$$\sin\theta = \frac{-\sigma_{11}}{\sqrt{\sigma_{11}^2 + \sigma_{12}^2}},$$
$$\cos\theta = \frac{\sigma_{12}}{\sqrt{\sigma_{11}^2 + \sigma_{12}^2}} \tag{6.56}$$

Substituting these into (6.55) yields

$$\tau_1 = \inf\left\{s > 0 : \sqrt{\sigma_{11}^2 + \sigma_{12}^2} \cdot Y_s^2 = 0\right\} \tag{6.57}$$

Clearly the quantity $\sqrt{\sigma_{11}^2 + \sigma_{12}^2}$ is strictly positive, thus

$$\tau_1 = \inf\left\{s > 0 : Y_s^2 = 0\right\}. \tag{6.58}$$

This is precisely what we should expect. Recall that $\tau_1$ represents the first time that process $Y_t$ hits line 1. However, line 1 has been the translated and rotated so that it is the x-axis. Thus $\tau_1$ represents the first time that the process $Y_t$ hits the x-axis (i.e. the first time that $Y_s^2 = 0$).

Our goal is to compute the price $E^Q\left[\frac{dP}{dQ}X\right]$ of the get-out option. We first write this out explicitly as

$$\begin{aligned} E^Q\left[\frac{dP}{dQ}X\right] = E^Q\Bigg[\frac{dP}{dQ}\Big(1_{A_1}&\left(e^{-r\tau_1}\cdot\left(S_{\tau_1}^2 - k_2\right)^+\right) \\ +1_{A_2}&\left(e^{-r\tau_2}\cdot\left(S_{\tau_2}^1 - k_1\right)^+\right) \\ +1_{A_3}&\left(e^{-rT}\left(\left(S_T^1 - k_1\right)^+ + \left(S_T^2 - k_2\right)^+\right)\right)\Big)\Bigg] \end{aligned} \tag{6.59}$$

The first term in this expectation is

$$E^Q\left[\frac{dP}{dQ}1_{A_1}\left(e^{-r\tau_1}\cdot\left(S_{\tau_1}^2 - k_2\right)^+\right)\right]. \tag{6.60}$$

Under the measure $Q$ we have

$$S_t^i = S^i \exp\left(\sigma_{i1}\tilde{W}_t^1 + \sigma_{i2}\tilde{W}_t^2\right) \text{ for i=1,2} \tag{6.61}$$

and

$$\frac{dP}{dQ} = \exp\left(\mu_1\tilde{W}_T^1 + \mu_2\tilde{W}_T^2 - \frac{\mu_1^2 + \mu_2^2}{2}T\right) \tag{6.62}$$

where $\tilde{W}_t^1$ and $\tilde{W}_t^2$ are standard Brownian motions. We now make the substitutions for $\tilde{W}_t^1$ and $\tilde{W}_t^2$ from (6.51). Thus (6.60) becomes

$$\begin{aligned} E^Q\Bigg[&\exp\left(\mu_1 \cdot f\left(Y_T^1, Y_T^2\right) + \mu_2 \cdot g\left(Y_T^1, Y_T^2\right) - \frac{\mu_1^2 + \mu_2^2}{2}T\right)\times \\ &1_{A_1}\left(e^{-r\tau_1} \cdot \left(S^2 \exp\left(\sigma_{21} \cdot f\left(Y_{\tau_1}^1, Y_{\tau_1}^2\right) + \sigma_{22} \cdot g\left(Y_{\tau_1}^1, Y_{\tau_1}^2\right)\right) - k_2\right)^+\right)\Bigg]. \end{aligned} \tag{6.63}$$

But $\tau_1 = \inf\left\{s > 0 : Y_s^2 = 0\right\}$ so $Y_{\tau_1}^2 = 0$. Replacing this above yields

$$\begin{aligned} E^Q\Bigg[&\exp\left(\mu_1 \cdot f\left(Y_T^1, Y_T^2\right) + \mu_2 \cdot g\left(Y_T^1, Y_T^2\right) - \frac{\mu_1^2 + \mu_2^2}{2}T\right)\times \\ &1_{A_1}\left(e^{-r\tau_1} \cdot \left(S^2 \exp\left(\sigma_{21} \cdot f\left(Y_{\tau_1}^1, 0\right) + \sigma_{22} \cdot g\left(Y_{\tau_1}^1, 0\right)\right) - k_2\right)^+\right)\Bigg]. \end{aligned} \tag{6.64}$$

We use some simple properties of expectations to take an inner expectation conditioned on the sigma algebra $\mathcal{F}_{T\wedge\tau_1}$. This yields

$$\begin{aligned} E^Q\Bigg[E\Bigg[&\exp\left(\mu_1 \cdot f\left(Y_T^1, Y_T^2\right) + \mu_2 \cdot g\left(Y_T^1, Y_T^2\right) - \frac{\mu_1^2 + \mu_2^2}{2}T\right)\times \\ &1_{A_1}\left(e^{-r\tau_1} \cdot \left(S^2 \exp\left(\sigma_{21} \cdot f\left(Y_{\tau_1}^1, 0\right) + \sigma_{22} \cdot g\left(Y_{\tau_1}^1, 0\right)\right) - k_2\right)^+\right)\Bigg|\mathcal{F}_{T\wedge\tau_1}\Bigg]\Bigg]. \end{aligned} \tag{6.65}$$

We take out the $\mathcal{F}_{T\wedge\tau_1}$ measurable portion inside of the inner expectation. All that is left is the Radon-Nikodym derivative, which is a martingale. Recall that $A_1 = \{\tau_1 < \tau_2, \tau_1 < T\}$. In particular, $\tau_1 < T$ on this set so $T \wedge \tau_1 = \tau_1$ on this set. Using the Optional Sampling theorem [7] for the process $Y_t$ and the bounded stopping time $T \wedge \tau_1$, (6.65) becomes

$$
\begin{aligned}
&E^Q\Bigg[1_{A_1}\left(e^{-r\tau_1}\cdot\left(S^2\exp\left(\sigma_{21}\cdot f\left(Y^1_{\tau_1},0\right)+\sigma_{22}\cdot g\left(Y^1_{\tau_1},0\right)\right)-k_2\right)^+\right)\times \\
&\qquad E\left[\exp\left(\mu_1\cdot f\left(Y^1_T,Y^2_T\right)+\mu_2\cdot g\left(Y^1_T,Y^2_T\right)-\frac{\mu_1^2+\mu_2^2}{2}T\right)\Big|\mathcal{F}_{T\wedge\tau_1}\right]\Bigg] \\
&= E^Q\Bigg[1_{A_1}\left(e^{-r\tau_1}\cdot\left(S^2\exp\left(\sigma_{21}\cdot f\left(Y^1_{\tau_1},0\right)+\sigma_{22}\cdot g\left(Y^1_{\tau_1},0\right)\right)-k_2\right)^+\right)\times \\
&\qquad \exp\left(\mu_1\cdot f\left(Y^1_{\tau_1},Y^2_{\tau_1}\right)+\mu_2\cdot g\left(Y^1_{\tau_1},Y^2_{\tau_1}\right)-\frac{\mu_1^2+\mu_2^2}{2}\tau_1\right)\Bigg]
\end{aligned}
\tag{6.66}
$$

We again observe that $Y^2_{\tau_1} = 0$ and substitute this into (6.66) to obtain

$$
\begin{aligned}
&E^Q\Bigg[1_{A_1}\left(e^{-r\tau_1}\cdot\left(S^2\exp\left(\sigma_{21}\cdot f\left(Y^1_{\tau_1},0\right)+\sigma_{22}\cdot g\left(Y^1_{\tau_1},0\right)\right)-k_2\right)^+\right)\times \\
&\qquad \exp\left(\mu_1\cdot f\left(Y^1_{\tau_1},0\right)+\mu_2\cdot g\left(Y^1_{\tau_1},0\right)-\frac{\mu_1^2+\mu_2^2}{2}\tau_1\right)\Bigg]
\end{aligned}
\tag{6.67}
$$

Observe that the random variable inside of the expectation depends only on $\tau_1$ and $Y^1_{\tau_1}$.

So we define a function $H(\cdot,\cdot)$ by

$$H(u,a)=e^{-ru}\cdot\left(S^2\exp\left(\sigma_{21}\cdot f(a,0)+\sigma_{22}\cdot g(a,0)\right)-k_2\right)\times \\ \exp\left(\mu_1\cdot f(a,0)+\mu_2\cdot g(a,0)-\frac{\mu_1^2+\mu_2^2}{2}u_1\right)\Bigg] \quad (6.68)$$

Note that the $(\cdot)^+$ notation has been removed. We may do this as long as we make sure that the quantity inside $(\cdot)^+$ is greater than or equal to zero. The following equivalent statements illustrate how to ensure this.

1) $S^2\exp\left(\sigma_{21}\cdot f(a,0)+\sigma_{22}\cdot g(a,0)\right)-k_2\geq 0$

2) $\sigma_{21}\cdot f(a,0)+\sigma_{22}\cdot g(a,0)\geq \tilde{k}_2=\ln\left(\frac{k_2}{S^2}\right)$

3) $\sigma_{21}\left(a\cos\theta+x_0\right)+\sigma_{22}\left(a\sin\theta+y_0\right)\geq \tilde{k}_2=\ln\left(\frac{k_2}{S^2}\right)$

4) $\sigma_{21}\cdot a\cos\theta+\sigma_{22}\cdot a\sin\theta+\left(\sigma_{21}x_0+\sigma_{22}y_0\right)\geq \tilde{k}_2$

5) $\sigma_{21}\cdot a\cos\theta+\sigma_{22}\cdot a\sin\theta+\tilde{b}\geq \tilde{k}_2$

6) $a\left(\sigma_{21}\cos\theta+\sigma_{22}\sin\theta\right)\geq \tilde{k}_2-\tilde{b}$

7) $a\left(\frac{\sigma_{12}\sigma_{21}-\sigma_{11}\sigma_{22}}{\sqrt{\sigma_{11}^2+\sigma_{12}^2}}\right)\geq \tilde{k}_2-\tilde{b}$

8) $a\geq\frac{\tilde{k}_2-\tilde{b}}{\sigma_{12}\sigma_{21}-\sigma_{11}\sigma_{22}}\sqrt{\sigma_{11}^2+\sigma_{12}^2}$

Statements 1-4 and 6 come from substituting and rearranging, 5 comes from a calculation similar to that of (6.54) and the coefficient on $a$ in 6 is positive so we may divide both sides by it to get to 7.

Define the constant

$$r_1 = \max\left(\frac{\tilde{k}_2 - \tilde{b}}{\sigma_{12}\sigma_{21} - \sigma_{11}\sigma_{22}}\sqrt{\sigma_{11}^2 + \sigma_{12}^2}, 0\right). \tag{6.69}$$

If $\tilde{k}_2 > \tilde{b}$, then we will only need to compute the expectation of $H\left(\tau_1, Y_{\tau_1}^1\right)$ over values of $Y_{\tau_1}^1$ which are greater than $\tilde{k}_2$. If $\tilde{k}_2 \le \tilde{b}$, then we will compute the expectation of $H\left(\tau_1, Y_{\tau_1}^1\right)$ over all values of $Y_{\tau_1}^1$, i.e. values greater than or equal to zero. So the constant $r_1$ is a lower bound for the values of $Y_{\tau_1}^1$ for which we must take the expectation. Thus we may write (6.67) as

$$E^Q\left[1_{A_1} 1_{\left\{Y_{\tau_1}^1 \ge r_1\right\}} H\left(\tau_1, Y_{\tau_1}^1\right)\right]. \tag{6.70}$$

But the joint density for the pair $\left(\tau_1, Y_{\tau_1}^1\right)$ when $\tau_1 < \tau_2$ given in [6] is

$$\begin{aligned} &P\left(\tau_1 \in du, Y_{\tau_1}^1 \in da, \tau_1 < \tau_2\right) \\ &= \frac{\pi}{\alpha^2 a u}\exp\left(-\frac{a^2 + r_0^2}{2u}\right)\sum_{j=0}^{+\infty}\left[j \cdot \sin\left(\frac{j\pi\theta_0}{\alpha}\right) I_{j\pi/\alpha}\left(\frac{a r_0}{t}\right)\right] \end{aligned} \tag{6.71}$$

where $\bar{p} = \begin{pmatrix} r_0 \cos\theta_0 \\ r_0 \sin\theta_0 \end{pmatrix}$, $\alpha = \pi - \tan^{-1}\left(-\frac{\sigma_{21}}{\sigma_{22}}\right) + \tan^{-1}\left(\frac{\sigma_{11}}{\sigma_{12}}\right)$ and $I_\beta$ is the modified Bessel function of order $\beta$. Recall that

$$I_\beta(x) = \left(\frac{x}{2}\right)^\beta \sum_{k=0}^{+\infty}\left[\left(\frac{x}{2}\right)^{2k} \frac{1}{k!\,\Gamma(\beta + k + 1)}\right] \tag{6.72}$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} dt$. Recall that $\alpha$ is simply the angle between the lines in which the process $Y_t$ starts. Thus we may express (6.70) as

$$\int_{0}^{T}\int_{r_1}^{+\infty} H(u,a)P\left(\tau_1 \in du, Y_{\tau_1}^1 \in da, \tau_1 < \tau_2\right)$$

$$= \int_{0}^{T}\int_{r_1}^{+\infty} H(u,a)\cdot\frac{\pi}{\alpha^2 a u}\exp\left(-\frac{a^2+r_0^2}{2u}\right)\times \qquad (6.73)$$

$$\sum_{j=0}^{+\infty}\left[j\cdot\sin\left(\frac{j\pi\theta_0}{\alpha}\right)I_{j\pi/\alpha}\left(\frac{ar_0}{t}\right)\right]da\,du$$

This representation allows for its precise value to be calculated via standard numerical integration techniques as opposed to simulation.

This is only the first term in the expectation. Fortunately, the same transformations enable us to use the densities in [6] to write out the expectations as Lebesgue integrals. The second expectation we must compute is

$$E\left[1_{A_2}\left(e^{-r\tau_2}\cdot\left(S_{\tau_2}^1-k_1\right)^+\right)\right]. \qquad (6.74)$$

We again use the functions $f$ and $g$ defined in (6.49) and (6.50) so that we have

$$\tilde{W}_t = \begin{pmatrix}\tilde{W}_t^1\\ \tilde{W}_t^2\end{pmatrix} = \begin{pmatrix} f\left(Y_t^1, Y_t^2\right)\\ g\left(Y_t^1, Y_t^2\right)\end{pmatrix}. \qquad (6.75)$$

We may then write (6.74) out as

$$E^Q\left[\exp\left(\mu_1\cdot f\left(Y_T^1,Y_T^2\right)+\mu_2\cdot g\left(Y_T^1,Y_T^2\right)-\frac{\mu_1^2+\mu_2^2}{2}T\right)\times\right.$$
$$\left.1_{A_2}\left(e^{-r\tau_2}\cdot\left(S^1\exp\left(\sigma_{11}\cdot f\left(Y_{\tau_2}^1,Y_{\tau_2}^2\right)+\sigma_{12}\cdot g\left(Y_{\tau_2}^1,Y_{\tau_2}^2\right)\right)-k_1\right)^+\right)\right]. \qquad (6.76)$$

We use the same method as in (6.66) so that (6.76) becomes

$$E^Q\left[\exp\left(\mu_1\cdot f\left(Y_{\tau_2}^1,Y_{\tau_2}^2\right)+\mu_2\cdot g\left(Y_{\tau_2}^1,Y_{\tau_2}^2\right)-\frac{\mu_1^2+\mu_2^2}{2}\tau_2\right)\times\right.$$
$$\left.1_{A_2}\left(e^{-r\tau_2}\cdot\left(S^1\exp\left(\sigma_{11}\cdot f\left(Y_{\tau_2}^1,Y_{\tau_2}^2\right)+\sigma_{12}\cdot g\left(Y_{\tau_2}^1,Y_{\tau_2}^2\right)\right)-k_1\right)^+\right)\right]. \qquad (6.77)$$

Recall that $\tau_2 = \inf\left\{s > 0 : \sigma_{21}\tilde{W}_s^1 + \sigma_{22}\tilde{W}_s^2 = \tilde{b}\right\}$. So

$$\begin{aligned}
\tau_2 &= \inf\left\{s > 0 : \sigma_{21}\tilde{W}_s^1 + \sigma_{22}\tilde{W}_s^2 = \tilde{b}\right\} \\
&= \inf\left\{s > 0 : \sigma_{21} \cdot f\left(Y_s^1, Y_s^2\right) + \sigma_{22} \cdot g\left(Y_s^1, Y_s^2\right) = \tilde{b}\right\} \\
&= \inf\left\{s > 0 : \sigma_{21}\left(Y_s^1 \cos\theta - Y_s^2 \sin\theta + x_0\right) + \sigma_{22}\left(Y_s^1 \sin\theta + Y_s^2 \cos\theta + x_0\right) = \tilde{b}\right\} \\
&= \inf\left\{s > 0 : \sigma_{21}\left(Y_s^1 \cos\theta - Y_s^2 \sin\theta + x_0\right) + \sigma_{22}\left(Y_s^1 \sin\theta + Y_s^2 \cos\theta + x_0\right) = \tilde{b}\right\} \\
&= \inf\left\{s > 0 : \left(\sigma_{21}\cos\theta + \sigma_{22}\sin\theta\right)Y_s^1 \right. \qquad (6.78) \\
&\qquad \left. + \left(\sigma_{22}\cos\theta - \sigma_{21}\sin\theta\right)Y_s^2 + \left(\sigma_{21}x_0 + \sigma_{22}y_0\right) = \tilde{b}\right\} \\
&= \inf\left\{s > 0 : \left(\sigma_{21}\cos\theta + \sigma_{22}\sin\theta\right)Y_s^1 + \left(\sigma_{22}\cos\theta - \sigma_{21}\sin\theta\right)Y_s^2 + \tilde{b} = \tilde{b}\right\} \\
&= \inf\left\{s > 0 : \left(\sigma_{21}\cos\theta + \sigma_{22}\sin\theta\right)Y_s^1 + \left(\sigma_{22}\cos\theta - \sigma_{21}\sin\theta\right)Y_s^2 = 0\right\} \\
&= \inf\left\{s > 0 : Y_s^1 = \frac{\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}} \cdot Y_s^2\right\}
\end{aligned}$$

The coefficient of the $Y_s^1$ term is precisely the reciprocal of the slope of line 2 after the transformation. Define another constant

$$m = \frac{\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}} \tag{6.79}$$

so that

$$\begin{aligned}
\tau_2 &= \inf\left\{s > 0 : Y_s^1 = \frac{\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}} \cdot Y_s^2\right\} \\
&= \inf\left\{s > 0 : Y_s^1 = m \cdot Y_s^2\right\}
\end{aligned} \tag{6.80}$$

Then we have

$$Y_{\tau_2}^1 = m \cdot Y_{\tau_2}^2 . \tag{6.81}$$

Now we may rewrite (6.77) as

$$E^Q\left[\exp\left(\mu_1\cdot f\left(m\cdot Y^2_{\tau_2},Y^2_{\tau_2}\right)+\mu_2\cdot g\left(m\cdot Y^2_{\tau_2},Y^2_{\tau_2}\right)-\frac{\mu_1^2+\mu_2^2}{2}\tau_2\right)\times\right.$$
$$\left.1_{A_2}\left(e^{-r\tau_2}\cdot\left(S^1\exp\left(\sigma_{11}\cdot f\left(m\cdot Y^2_{\tau_2},Y^2_{\tau_2}\right)+\sigma_{12}\cdot g\left(m\cdot Y^2_{\tau_2},Y^2_{\tau_2}\right)\right)-k_1\right)^+\right)\right] \tag{6.82}$$

so that the random variable we are taking the expectation of depends only on $\tau_2$ and $Y^2_{\tau_2}$.

Define a function $M(\cdot,\cdot)$ by

$$M(u,a)=\exp\left(\mu_1\cdot f(m\cdot a,a)+\mu_2\cdot g(m\cdot a,a)-\frac{\mu_1^2+\mu_2^2}{2}u\right)\times$$
$$\left(e^{-ru}\cdot\left(S^1\exp\left(\sigma_{11}\cdot f(m\cdot a,a)+\sigma_{12}\cdot g(m\cdot a,a)\right)-k_1\right)\right). \tag{6.83}$$

Then we must again find the lower bound $r_2$ for the random variable $Y^2_{\tau_2}$ so that we may omit the $(\cdot)^+$ notation. Computations similar to those for $r_1$ yield

$$r_2=\max\left(\frac{\tilde{k}_1-\tilde{a}}{\sqrt{\sigma_{11}^2+\sigma_{12}^2}},0\right). \tag{6.84}$$

So we may express (6.82) as

$$E^Q\left[1_{A_2}1_{\left\{Y^2_{\tau_2}\geq r_2\right\}}M\left(\tau_2,Y^2_{\tau_2}\right)\right]. \tag{6.85}$$

Thus we only need the joint distribution for the pair $\left(\tau_2,Y^2_{\tau_2}\right)$ given $\tau_2<\tau_1$. But this is given in [6] by

$$P\left(\tau_2 \in du, Y^2_{\tau_2} \in da, \tau_2 < \tau_1\right)$$
$$= \frac{\pi \cdot \cos\left(\alpha - \pi/2\right)}{\alpha^2 a\, u} \exp\left( - \frac{a^2 + r_0^2 \cdot \cos^2\left(\alpha - \pi/2\right)}{2u \cdot \cos^2\left(\alpha - \pi/2\right)} \right) \times \qquad (6.86)$$
$$\sum_{j=0}^{+\infty} \left[ -1^j \cdot j \cdot \sin\left(\frac{j\pi\tilde{\theta}_0}{\alpha}\right) I_{j\pi/\alpha}\left(\frac{a r_0}{\cos\left(\alpha - \pi/2\right) t}\right)\right]$$

where $\tilde{\theta}_0 = \alpha - \theta_0$. Note: If $\vec{q}$ is the point we get by reflecting $\vec{p} = \begin{pmatrix} r_0 \cos\theta_0 \\ r_0 \sin\theta_0 \end{pmatrix}$ about the line $y = \tan\left(\frac{\alpha}{2}\right) \cdot x$, then $\vec{q} = \begin{pmatrix} r_0 \cos\tilde{\theta}_0 \\ r_0 \sin\tilde{\theta}_0 \end{pmatrix}$. The $\cos\left(\alpha - \pi/2\right)$ factor appears as a result of projecting the process $Y_t$ onto the y-axis to obtain the random variable $Y^2_{\tau_2}$.

**Remark:** *The formula in [6] is given in polar coordinates and thus requires this additional transformation. Note that this was not necessary for the case of the first expectation. That is due to the fact that the polar coordinate radial value is identical to that of $Y^1_{\tau_1}$ since $Y^1_{\tau_1}$ was on the x-axis.*

Using the density above we may express (6.82) as

$$E^Q\left[ 1_{A_2} 1_{\left\{Y^2_{\tau_2} \ge r_2\right\}} M\left(\tau_2, Y^2_{\tau_2}\right)\right]$$
$$= \int_0^T \int_{r_2}^{+\infty} M(u,a) P\left(\tau_2 \in du, Y^2_{\tau_2} \in da, \tau_2 < \tau_1\right)$$
$$= \int_0^T \int_{r_2}^{+\infty} M(u,a) \cdot \frac{\pi \cdot \cos\left(\alpha - \pi/2\right)}{\alpha^2 a\, u} \exp\left( - \frac{a^2 + r_0^2 \cdot \cos^2\left(\alpha - \pi/2\right)}{2u \cdot \cos^2\left(\alpha - \pi/2\right)} \right) \times \qquad (6.87)$$
$$\sum_{j=0}^{+\infty} \left[ -1^j \cdot j \cdot \sin\left(\frac{j\pi\tilde{\theta}_0}{\alpha}\right) I_{j\pi/\alpha}\left(\frac{a r_0}{\cos\left(\alpha - \pi/2\right) t}\right)\right] da\, du$$

We need only compute the final expectation now. Just as with the first two terms, we will use the transformation so that we can use another density provided in [6]. Recall that we are computing

$$E^Q\left[\frac{dP}{dQ}1_{A_3}\left(e^{-rT}\left(\left(S_T^1-k_1\right)^+ + \left(S_T^2-k_2\right)^+\right)\right)\right]. \tag{6.88}$$

We break this up into two separate expectations

$$e^{-rT}\cdot E^Q\left[\frac{dP}{dQ}1_{A_3}\left(S_T^1-k_1\right)^+\right]+e^{-rT}E^Q\left[\frac{dP}{dQ}1_{A_3}\left(S_T^2-k_2\right)^+\right]. \tag{6.89}$$

We focus on the first expectation. First we remove the $(\cdot)^+$ notation with the use of an appropriate indicator function. This yields

$$e^{-rT}\cdot E^Q\left[\frac{dP}{dQ}1_{A_3}1_{\{S_T^1-k_1\geq 0\}}\left(S_T^1-k_1\right)\right] \tag{6.90}$$

Now we express the random variable in the expectation in terms of the transformed process $Y_t$. The indicator function $1_{\{S_T^1-k_1\geq 0\}}$ is left as is but will be changed momentarily.

This yields

$$\begin{aligned} E^Q\Bigg[&\exp\left(\mu_1\cdot f\left(Y_T^1,Y_T^2\right)+\mu_2\cdot g\left(Y_T^1,Y_T^2\right)-\frac{\mu_1^2+\mu_2^2}{2}T\right)\times \\ &1_{A_3}1_{\{S_T^1-k_1\geq 0\}}\left(e^{-rT}\cdot\left(S^1\exp\left(\sigma_{11}\cdot f\left(Y_T^1,Y_T^2\right)+\sigma_{12}\cdot g\left(Y_T^1,Y_T^2\right)\right)-k_1\right)\right)\Bigg]\end{aligned} \tag{6.91}$$

Again we simplify the notation by defining the function $L(\cdot,\cdot)$ by

$$\begin{aligned} L(x,y)=e^{-rT}\exp&\left(\mu_1\cdot f(x,y)+\mu_2\cdot g(x,y)-\frac{\mu_1^2+\mu_2^2}{2}T\right)\times \\ &\left(S^1\exp\left(\sigma_{11}\cdot f(x,y)+\sigma_{12}\cdot g(x,y)\right)-k_1\right)\end{aligned}. \tag{6.92}$$

Then we may write (6.91) as

$$E^Q\left[1_{A_3}1_{\{S_T^1-k_1\geq 0\}}L\left(Y_T^1,Y_T^2\right)\right]. \tag{6.93}$$

The random variable in this expectation depends only on the process $Y_T$ when $A_3=\{\tau_1\geq T;\ \ \tau_2\geq T\}$. But we have this density as well from [6]. The density is given in polar coordinates as

$$\begin{aligned}&P(\tau_1>T,\tau_2>T,Y_T\in dy)\\&\quad=\frac{2r}{T\alpha}\exp\left(-\frac{r^2+r_0^2}{2T}\right)\sum_{j=0}^{+\infty}\sin\frac{j\pi\phi}{\alpha}\sin\frac{j\pi\theta_0}{\alpha}I_{j\pi/\alpha}\left(\frac{rr_0}{T}\right)dr\,d\phi\end{aligned}. \tag{6.94}$$

Just as before, we need to examine the indicator function a little closer in order to determine the region over which we will integrate.

$$\begin{aligned}\left\{S_T^1-k_1\geq 0\right\}&=\left\{S^1\exp\left(\sigma_{11}\cdot f\left(Y_T^1,Y_T^2\right)+\sigma_{12}\cdot g\left(Y_T^1,Y_T^2\right)\right)-k_1\geq 0\right\}\\&=\left\{\sigma_{11}\cdot f\left(Y_T^1,Y_T^2\right)+\sigma_{12}\cdot g\left(Y_T^1,Y_T^2\right)\geq\tilde{k}_1\right\}\end{aligned} \tag{6.95}$$

where $\tilde{k}_1=\ln\left(\frac{k_1}{S^1}\right)$ as before. If we write (6.93) as a Lebesgue integral we have

$$\begin{aligned}&E^Q\left[1_{A_3}1_{\{S_T^1-k_1\geq 0\}}L\left(Y_T^1,Y_T^2\right)\right]\\&\quad=\int_0^{\alpha}\int_0^{+\infty}1_B L(r\cos\theta,r\sin\theta)P(\tau_1>T,\tau_2>T,Y_t\in dr\,d\theta)\end{aligned} \tag{6.96}$$

where $B=\left\{\sigma_{11}\cdot f(r\cos\theta,r\sin\theta)+\sigma_{12}\cdot g(r\cos\theta,r\sin\theta)\geq\tilde{k}_1\right\}$. We may rewrite $B$ as

$$
\begin{aligned}
B &= \{\sigma_{11}(r\cos\phi\cdot\cos\theta - r\sin\phi\cdot\sin\theta + x_0) \\
&\qquad +\sigma_{12}(r\cos\phi\sin\theta + r\sin\phi\cos\theta + y_0) \geq \tilde{k}_1 \\
&= \{(r\cdot\sigma_{11}\cos\theta + r\cdot\sigma_{12}\sin\theta)\cos\phi \\
&\qquad +(r\cdot\sigma_{12}\cos\theta - r\cdot\sigma_{11}\sin\theta)\sin\phi \\
&\qquad\qquad +r(\sigma_{11}x_0 + \sigma_{12}y_0) \geq \tilde{k}_1\} \\
&= \{(r\cdot\sigma_{12}\cos\theta - r\cdot\sigma_{11}\sin\theta)\sin\phi + \tilde{a} \geq \tilde{k}_1\} \\
&= \{r(\sigma_{12}\cos\theta - \sigma_{11}\sin\theta)\sin\phi \geq \tilde{k}_1 - \tilde{a}\} \\
&= \left\{r\sqrt{\sigma_{11}^2 + \sigma_{12}^2}\sin\phi \geq \tilde{k}_1 - \tilde{a}\right\} \\
&= \left\{r \geq \frac{(\tilde{k}_1 - \tilde{a})}{\sqrt{\sigma_{11}^2 + \sigma_{12}^2}\cdot\sin\phi}\right\}
\end{aligned}
\tag{6.97}
$$

Define the function $q_1(\phi)$ by

$$
q_1(\phi) = \max\left(\frac{\tilde{k}_1 - \tilde{a}}{\sqrt{\sigma_{11}^2 + \sigma_{12}^2}\cdot\sin\phi}, 0\right) \tag{6.98}
$$

so that the lower bound on $r$ for a fixed value of $\phi$ is $q_1(\phi)$. The maximum between the quantity above and zero is used to ensure that $r \geq 0$ in the case when $k_1 < a$ which is equivalent to $\tilde{k}_1 < \tilde{a}$. Now we may express (6.93) as

$$
\begin{aligned}
&E^Q\left[1_{A_3}1_{\{S_T^1 - k_1 \geq 0\}}L(Y_T^1, Y_T^2)\right] \\
&\quad = \int_0^{\alpha}\int_{q_1(\phi)}^{+\infty} L(r\cos\phi, r\sin\phi)P(\tau_1 > T, \tau_2 > T, Y_t \in dr\,d\phi)
\end{aligned}
\tag{6.99}
$$

We compute the second expectation in just the same manner. We write out the expectation in terms of the transformed process. Recall the second expectation is

$$
e^{-rT}E^Q\left[\frac{dP}{dQ}1_{A_3}\left(S_T^2 - k_2\right)^+\right]. \tag{6.100}
$$

Written out in terms of the transformed process $Y_t$ this looks like

$$E^Q\left[\exp\left(\mu_1\cdot f\left(Y_T^1,Y_T^2\right)+\mu_2\cdot g\left(Y_T^1,Y_T^2\right)-\frac{\mu_1^2+\mu_2^2}{2}T\right)\times \right. \\ \left. 1_{A_3}1_{\{S_T^2-k_2\geq 0\}}\left(e^{-rT}\cdot\left(S^2\exp\left(\sigma_{21}\cdot f\left(Y_T^1,Y_T^2\right)+\sigma_{22}\cdot g\left(Y_T^1,Y_T^2\right)\right)-k_2\right)\right)\right]. \tag{6.101}$$

Define the function $U(\cdot,\cdot)$ by

$$U(x,y)=\exp\left(\mu_1\cdot f(x,y)+\mu_2\cdot g(x,y)-\frac{\mu_1^2+\mu_2^2}{2}T\right)\times \\ \left(e^{-rT}\cdot\left(S^2\exp\left(\sigma_{21}\cdot f(x,y)+\sigma_{22}\cdot g(x,y)\right)-k_2\right)\right). \tag{6.102}$$

Then (6.101) becomes

$$E^Q\left[1_{A_3}1_{\{S_T^2-k_2\geq 0\}}U\left(Y_T^1,Y_T^2\right)\right]. \tag{6.103}$$

Again the random variable depends only on the process $Y_t$ when $A_3=\{\tau_1\geq T;\ \tau_2\geq T\}$.

So we may express this in terms of the Lebesgue integral

$$E^Q\left[1_{A_3}1_{\{S_T^2-k_2\geq 0\}}U\left(Y_T^1,Y_T^2\right)\right] \\ =\int_0^{\alpha}\int_0^{+\infty}1_E U(r\cos\phi,r\sin\phi)P\left(\tau_1>T,\tau_2>T,Y_t\in dr\,d\phi\right) \tag{6.104}$$

where $E=\{\sigma_{21}\cdot f(r\cos\phi,r\sin\phi)+\sigma_{22}\cdot g(r\cos\phi,r\sin\phi)\geq\tilde{k}_2\}$. Calculations similar to those of (6.97) show that

$$E=\left\{\left(\tilde{k}_2-\tilde{b}\right)\cdot\left[\left(\frac{\sigma_{12}\sigma_{21}-\sigma_{11}\sigma_{22}}{\sqrt{\sigma_{11}^2+\sigma_{12}^2}}\right)\cos\phi+\left(\frac{\sigma_{12}\sigma_{22}+\sigma_{11}\sigma_{21}}{\sqrt{\sigma_{11}^2+\sigma_{12}^2}}\right)\sin\phi\right]^{-1}\right\}. \tag{6.105}$$

Define the function $q_2(\cdot)$ by

$$q_2(\phi) = \max\left( (\tilde{k}_2 - \tilde{b}) \cdot \left[ \left( \frac{\sigma_{12}\sigma_{21} - \sigma_{11}\sigma_{22}}{\sqrt{\sigma_{11}^2 + \sigma_{12}^2}} \right) \cos\phi + \left( \frac{\sigma_{12}\sigma_{22} + \sigma_{11}\sigma_{21}}{\sqrt{\sigma_{11}^2 + \sigma_{12}^2}} \right) \sin\phi \right]^{-1}, 0 \right) \tag{6.106}$$

so that the lower bound on $r$ for a fixed value of $\phi$ is $q_2(\phi)$. Then we may express

(6.104) as

$$E^Q\left[ 1_{A_3} 1_{\{S_T^2 - k_2 \geq 0\}} U\left(Y_T^1, Y_T^2\right) \right] = \int_0^{\alpha} \int_{q_2(\phi)}^{+\infty} U(r\cos\phi, r\sin\phi) P(\tau_1 > T, \tau_2 > T, Y_t \in dr\, d\phi) . \tag{6.107}$$

This being the last necessary calculation, we may add up the quantities in (6.73), (6.87),

(6.99) and (6.107) in order to obtain the price of the get-out option.

## APPENDIX
## EXPLICIT OPTIONS' PRICING FORMULAS

The results for the partial tunnel options' prices from chapter 4 were expressed in terms of the functions $h(c,d)$, $\tilde{h}(c,d)$ and $w(c,d)$ from chapter 3. Here we present the options' prices explicitly in terms of the bivariate normal distribution function. We first make some computations that will be used later.

Recall that $\mu = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)$. So the quantity $\frac{(\mu+\sigma)^2 T}{2}$ becomes

$$\begin{aligned}
\frac{(\mu+\sigma)^2 T}{2} &= \frac{\left(\mu^2 + 2\sigma\mu + \sigma^2\right)T}{2} \\
&= \frac{\mu^2 T}{2} + \frac{\left(2\sigma\mu + \sigma^2\right)T}{2} \\
&= \frac{\mu^2 T}{2} + \frac{\left(2\sigma \cdot \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right) + \sigma^2\right)T}{2} \\
&= \frac{\mu^2 T}{2} + \frac{\left(2r - \sigma^2 + \sigma^2\right)T}{2} \\
&= \frac{\mu^2 T}{2} + rT
\end{aligned}$$

As a result of theorem 4.2 we have

$$V_1^C = \sum_{j=-\infty}^{+\infty} e^{-rT-\frac{1}{2}\mu^2 T}\left[S\cdot\left(h(\mu+\sigma,c_j) - h(\mu+\sigma,d_j)\right) - k\cdot\left(h(\mu,c_j) - h(\mu,d_j)\right)\right] \quad \text{(A.1)}$$

where

$$h(c,d) = \exp\left(\frac{c^2T}{2} - cd\right)\left[N_\rho\left(\frac{\alpha-\tilde{b}}{\sqrt{t_1}}, \frac{\beta-\tilde{k}}{\sqrt{T}}\right) - N_\rho\left(\frac{\alpha-\tilde{a}}{\sqrt{t_1}}, \frac{\beta-\tilde{k}}{\sqrt{T}}\right)\right], \tag{A.2}$$

$c_j = 2(\tilde{b}-\tilde{a})$, $d_j = 2(\tilde{b}-\tilde{a}) - 2\tilde{b}$, $\rho = \sqrt{\frac{t_1}{T}}$, $\alpha = ct_1 - d$ and $\beta = cT - d$. Substituting $h(c,d)$ and simplifying yields

$$\begin{aligned}
V_1^C = \sum_{j=-\infty}^{+\infty}\Bigg[S\Bigg(& e^{-(\mu+\sigma)c_j}\left[N_\rho\left(\frac{(\mu+\sigma)t_1 - c_j - \tilde{a}}{\sqrt{t_1}}, \frac{(\mu+\sigma)T - c_j - \tilde{k}}{\sqrt{T}}\right)\right. \\
&\qquad \left. - N_\rho\left(\frac{(\mu+\sigma)t_1 - c_j - \tilde{b}}{\sqrt{t_1}}, \frac{(\mu+\sigma)T - c_j - \tilde{k}}{\sqrt{T}}\right)\right] \\
&- e^{-(\mu+\sigma)d_j}\left[N_\rho\left(\frac{(\mu+\sigma)t_1 - d_j - \tilde{a}}{\sqrt{t_1}}, \frac{(\mu+\sigma)T - d_j - \tilde{k}}{\sqrt{T}}\right)\right. \\
&\qquad \left. - N_\rho\left(\frac{(\mu+\sigma)t_1 - d_j - \tilde{b}}{\sqrt{t_1}}, \frac{(\mu+\sigma)T - d_j - \tilde{k}}{\sqrt{T}}\right)\right]\Bigg) \\
- e^{-rT}k\cdot\Bigg(& e^{-\mu c_j}\left[N_\rho\left(\frac{\mu t_1 - c_j - \tilde{a}}{\sqrt{t_1}}, \frac{\mu T - c_j - \tilde{k}}{\sqrt{T}}\right)\right. \\
&\qquad \left. - N_\rho\left(\frac{\mu t_1 - c_j - \tilde{b}}{\sqrt{t_1}}, \frac{\mu T - c_j - \tilde{k}}{\sqrt{T}}\right)\right] \\
&- e^{-\mu d_j}\left[N_\rho\left(\frac{\mu t_1 - d_j - \tilde{a}}{\sqrt{t_1}}, \frac{\mu T - d_j - \tilde{k}}{\sqrt{T}}\right)\right. \\
&\qquad \left. - N_\rho\left(\frac{\mu t_1 - d_j - \tilde{b}}{\sqrt{t_1}}, \frac{\mu T - d_j - \tilde{k}}{\sqrt{T}}\right)\right]\Bigg)\Bigg]
\end{aligned}$$

Theorem 4.4 gave us the price $V_1^P$ of the PTPO-I as

$V_1^P = \sum_{j=-\infty}^{+\infty} e^{-rT-\frac{1}{2}\mu^2 T}\left[k\cdot\left(\hat{h}(\mu,c_j) - \hat{h}(\mu,d_j)\right) - S\cdot\left(\hat{h}(\mu+\sigma,c_j) - \hat{h}(\mu+\sigma,d_j)\right)\right]$ where

$$\hat{h}(c,d)=\exp\left(\frac{c^2T}{2}-cd\right)\left[N_\rho\left(\frac{\tilde{b}-\alpha}{\sqrt{t_1}},\frac{\tilde{k}-\beta}{\sqrt{T}}\right)-N_\rho\left(\frac{\tilde{a}-\alpha}{\sqrt{t_1}},\frac{\tilde{k}-\beta}{\sqrt{T}}\right)\right],$$

$c_j=2(\tilde{b}-\tilde{a})$, $d_j=2(\tilde{b}-\tilde{a})-2\tilde{b}$, $\rho=\sqrt{\frac{t_1}{T}}$, $\alpha=ct_1-d$ and $\beta=cT-d$.

Substituting $\tilde{h}(c,d)$ and simplifying yields

$$\begin{aligned}
V_1^P &= \\
&\sum_{j=-\infty}^{+\infty}\Bigg[e^{-rT}k\cdot\Bigg(e^{-\mu c_j}\left[N_\rho\left(\frac{\tilde{b}-\mu t_1+c_j}{\sqrt{t_1}},\frac{\tilde{k}-\mu t_1+c_j}{\sqrt{T}}\right)-N_\rho\left(\frac{\tilde{a}-\mu T+c_j}{\sqrt{t_1}},\frac{\tilde{k}-\mu t_1+c_j}{\sqrt{T}}\right)\right]\\
&\qquad -e^{-\mu d_j}\left[N_\rho\left(\frac{\tilde{b}-\mu t_1+d_j}{\sqrt{t_1}},\frac{\tilde{k}-\mu t_1+d_j}{\sqrt{T}}\right)-N_\rho\left(\frac{\tilde{a}-\mu T+d_j}{\sqrt{t_1}},\frac{\tilde{k}-\mu t_1+d_j}{\sqrt{T}}\right)\right]\Bigg)\\
&\quad -S\cdot\Bigg(e^{-(\mu+\sigma)c_j}\Bigg[N_\rho\left(\frac{\tilde{b}-(\mu+\sigma)t_1+c_j}{\sqrt{t_1}},\frac{\tilde{k}-(\mu+\sigma)T+c_j}{\sqrt{T}}\right)\\
&\qquad -N_\rho\left(\frac{\tilde{a}-(\mu+\sigma)t_1+c_j}{\sqrt{t_1}},\frac{\tilde{k}-(\mu+\sigma)T+c_j}{\sqrt{T}}\right)\Bigg]\\
&\quad -e^{-(\mu+\sigma)d_j}\Bigg[N_\rho\left(\frac{\tilde{b}-(\mu+\sigma)t_1+d_j}{\sqrt{t_1}},\frac{\tilde{k}-(\mu+\sigma)T+d_j}{\sqrt{T}}\right)\\
&\qquad -N_\rho\left(\frac{\tilde{a}-(\mu+\sigma)t_1+d_j}{\sqrt{t_1}},\frac{\tilde{k}-(\mu+\sigma)T+d_j}{\sqrt{T}}\right)\Bigg]\Bigg)\Bigg]
\end{aligned}$$

Theorem 4.5 gave use the price $V_{\text{II}}^{\text{C}}$ of the PTCO-II as

$$V_{\text{II}}^{\text{C}}=\sum_{j=-\infty}^{+\infty}e^{-rT-\frac{1}{2}\mu^2T}\left[S\cdot\left(w(\mu+\sigma,c_j)-w(\mu+\sigma,d_j)\right)-k\cdot\left(w(\mu,c_j)-w(\mu,d_j)\right)\right]$$

where

$$w(c,d) = \exp\left(\frac{c^2T}{2} - cd\right)\left(\left[N_\rho\left(\frac{\tilde{b}+d-\alpha}{\sqrt{t_1}}, \frac{\tilde{b}-\beta}{\sqrt{T}}\right) - N_\rho\left(\frac{\tilde{a}+d-\alpha}{\sqrt{t_1}}, \frac{\tilde{b}-\beta}{\sqrt{T}}\right)\right]\right.$$
$$\left. - \left[N_\rho\left(\frac{\tilde{b}+d-\alpha}{\sqrt{t_1}}, \frac{\delta-\beta}{\sqrt{T}}\right) - N_\rho\left(\frac{\tilde{a}+d-\alpha}{\sqrt{t_1}}, \frac{\delta-\beta}{\sqrt{T}}\right)\right]\right),$$

$\delta = \max(\tilde{a}, \tilde{k})$, $c_j = 2(\tilde{b} - \tilde{a})$, $d_j = 2(\tilde{b} - \tilde{a}) - 2\tilde{b}$, $\alpha = ct_1 - d$ and $\beta = cT - d$.

Substituting $w(c,d)$ and simplifying yields

$$V_{\text{II}}^{C} =$$
$$\sum_{j=-\infty}^{+\infty}\left[S\cdot\left(e^{-\mu' c_j}\left(\left[N_\rho\left(\frac{\tilde{b}+2c_j-\mu' t_1}{\sqrt{t_1}}, \frac{\tilde{b}-\mu' T+c_j}{\sqrt{T}}\right) - N_\rho\left(\frac{\tilde{a}+2c_j-\mu' t_1}{\sqrt{t_1}}, \frac{\tilde{b}-\mu' T+c_j}{\sqrt{T}}\right)\right]\right.\right.\right.$$
$$\left. - \left[N_\rho\left(\frac{\tilde{b}+2c_j-\mu' t_1}{\sqrt{t_1}}, \frac{\delta-\mu' T+c_j}{\sqrt{T}}\right) - N_\rho\left(\frac{\tilde{a}+2c_j-\mu' t_1}{\sqrt{t_1}}, \frac{\delta-\mu' T+c_j}{\sqrt{T}}\right)\right]\right)$$
$$- e^{-\mu' d_j}\left(\left[N_\rho\left(\frac{\tilde{b}+2d_j-\mu' t_1}{\sqrt{t_1}}, \frac{\tilde{b}-\mu' T+d_j}{\sqrt{T}}\right) - N_\rho\left(\frac{\tilde{a}+2d_j-\mu' t_1}{\sqrt{t_1}}, \frac{\tilde{b}-\mu' T+d_j}{\sqrt{T}}\right)\right]\right.$$
$$\left.\left. - \left[N_\rho\left(\frac{\tilde{b}+2d_j-\mu' t_1}{\sqrt{t_1}}, \frac{\delta-\mu' T+d_j}{\sqrt{T}}\right) - N_\rho\left(\frac{\tilde{a}+2d_j-\mu' t_1}{\sqrt{t_1}}, \frac{\delta-\mu' T+d_j}{\sqrt{T}}\right)\right]\right)\right)$$
$$- e^{-rT}k\cdot\left(e^{-\mu c_j}\left(\left[N_\rho\left(\frac{\tilde{b}+2c_j-\mu t_1}{\sqrt{t_1}}, \frac{\tilde{b}-\mu T+c_j}{\sqrt{T}}\right) - N_\rho\left(\frac{\tilde{a}+2c_j-\mu t_1}{\sqrt{t_1}}, \frac{\tilde{b}-\mu T+c_j}{\sqrt{T}}\right)\right]\right.\right.$$
$$\left. - \left[N_\rho\left(\frac{\tilde{b}+2c_j-\mu t_1}{\sqrt{t_1}}, \frac{\delta-\mu T+c_j}{\sqrt{T}}\right) - N_\rho\left(\frac{\tilde{a}+2c_j-\mu t_1}{\sqrt{t_1}}, \frac{\delta-\mu T+c_j}{\sqrt{T}}\right)\right]\right)$$
$$- e^{-\mu d_j}\left(\left[N_\rho\left(\frac{\tilde{b}+2d_j-\mu t_1}{\sqrt{t_1}}, \frac{\tilde{b}-\mu T+d_j}{\sqrt{T}}\right) - N_\rho\left(\frac{\tilde{a}+2d_j-\mu t_1}{\sqrt{t_1}}, \frac{\tilde{b}-\mu T+d_j}{\sqrt{T}}\right)\right]\right.$$
$$\left.\left.\left. - \left[N_\rho\left(\frac{\tilde{b}+2d_j-\mu t_1}{\sqrt{t_1}}, \frac{\delta-\mu T+d_j}{\sqrt{T}}\right) - N_\rho\left(\frac{\tilde{a}+2d_j-\mu t_1}{\sqrt{t_1}}, \frac{\delta-\mu T+d_j}{\sqrt{T}}\right)\right]\right)\right)\right]$$

where $\mu' = \mu + \sigma$.

Theorem 4.6 gave use the price $V_{\text{II}}^{P}$ of the PTPO-II as

$$V_{\text{II}}^{P} = \sum_{j=-\infty}^{+\infty} e^{-rT-\frac{1}{2}\mu^2 T}\left[k\cdot\left(w(\mu,c_j) - w(\mu,d_j)\right) - S\cdot\left(w(\mu+\sigma,c_j) - w(\mu+\sigma,d_j)\right)\right]$$

where

$$w(c,d)=\exp\left(\frac{c^2T}{2}-cd\right)\left(\left[N_\rho\left(\frac{\tilde{b}+d-\alpha}{\sqrt{t_1}},\frac{\psi-\beta}{\sqrt{T}}\right)-N_\rho\left(\frac{\tilde{a}+d-\alpha}{\sqrt{t_1}},\frac{\psi-\beta}{\sqrt{T}}\right)\right]\right.$$
$$\left.-\left[N_\rho\left(\frac{\tilde{b}+d-\alpha}{\sqrt{t_1}},\frac{\tilde{a}-\beta}{\sqrt{T}}\right)-N_\rho\left(\frac{\tilde{a}+d-\alpha}{\sqrt{t_1}},\frac{\tilde{a}-\beta}{\sqrt{T}}\right)\right]\right),$$

$\psi=\min(\tilde{b},\tilde{k})$, $c_j=2(\tilde{b}-\tilde{a})$, $d_j=2(\tilde{b}-\tilde{a})-2\tilde{b}$, $\alpha=ct_1-d$ and $\beta=cT-d$.

Substituting $w(c,d)$ and simplifying

yields

$$V_{\mathrm{II}}^{P}=$$
$$\sum_{j=-\infty}^{+\infty}\left[e^{-rT}k\cdot\left(e^{-\mu c_j}\left(\left[N_\rho\left(\frac{\tilde{b}+2c_j-\mu t_1}{\sqrt{t_1}},\frac{\psi-\mu T+c_j}{\sqrt{T}}\right)-N_\rho\left(\frac{\tilde{a}+2c_j-\mu t_1}{\sqrt{t_1}},\frac{\psi-\mu T+c_j}{\sqrt{T}}\right)\right]\right.\right.\right.$$
$$\left.-\left[N_\rho\left(\frac{\tilde{b}+2c_j-\mu t_1}{\sqrt{t_1}},\frac{\tilde{b}-\mu T+c_j}{\sqrt{T}}\right)-N_\rho\left(\frac{\tilde{a}+2c_j-\mu t_1}{\sqrt{t_1}},\frac{\tilde{b}-\mu T+c_j}{\sqrt{T}}\right)\right]\right)$$
$$-e^{-\mu d_j}\left(\left[N_\rho\left(\frac{\tilde{b}+2d_j-\mu t_1}{\sqrt{t_1}},\frac{\psi-\mu T+d_j}{\sqrt{T}}\right)-N_\rho\left(\frac{\tilde{a}+2d_j-\mu t_1}{\sqrt{t_1}},\frac{\psi-\mu T+d_j}{\sqrt{T}}\right)\right]\right.$$
$$\left.\left.-\left[N_\rho\left(\frac{\tilde{b}+2d_j-\mu t_1}{\sqrt{t_1}},\frac{\tilde{b}-\mu T+d_j}{\sqrt{T}}\right)-N_\rho\left(\frac{\tilde{a}+2d_j-\mu t_1}{\sqrt{t_1}},\frac{\tilde{b}-\mu T+d_j}{\sqrt{T}}\right)\right]\right)\right)$$
$$-S\cdot\left(e^{-\mu' c_j}\left(\left[N_\rho\left(\frac{\tilde{b}+2c_j-\mu' t_1}{\sqrt{t_1}},\frac{\psi-\mu' T+c_j}{\sqrt{T}}\right)-N_\rho\left(\frac{\tilde{a}+2c_j-\mu' t_1}{\sqrt{t_1}},\frac{\psi-\mu' T+c_j}{\sqrt{T}}\right)\right]\right.\right.$$
$$\left.-\left[N_\rho\left(\frac{\tilde{b}+2c_j-\mu' t_1}{\sqrt{t_1}},\frac{\tilde{b}-\mu' T+c_j}{\sqrt{T}}\right)-N_\rho\left(\frac{\tilde{a}+2c_j-\mu' t_1}{\sqrt{t_1}},\frac{\tilde{b}-\mu' T+c_j}{\sqrt{T}}\right)\right]\right)$$
$$-e^{-\mu' d_j}\left(\left[N_\rho\left(\frac{\tilde{b}+2d_j-\mu' t_1}{\sqrt{t_1}},\frac{\psi-\mu' T+d_j}{\sqrt{T}}\right)-N_\rho\left(\frac{\tilde{a}+2d_j-\mu' t_1}{\sqrt{t_1}},\frac{\psi-\mu' T+d_j}{\sqrt{T}}\right)\right]\right.$$
$$\left.\left.\left.-\left[N_\rho\left(\frac{\tilde{b}+2d_j-\mu' t_1}{\sqrt{t_1}},\frac{\tilde{b}-\mu' T+d_j}{\sqrt{T}}\right)-N_\rho\left(\frac{\tilde{a}+2d_j-\mu' t_1}{\sqrt{t_1}},\frac{\tilde{b}-\mu' T+d_j}{\sqrt{T}}\right)\right]\right)\right)\right]$$

## REFERENCES


[1] Baxter, Martin and Andrew Rennie, *Financial Calculus*, Cambridge University Press, New York, 1996.

[2] Chuang, Chin-Shan, Joint distribution of Brownian motion and its maximum, *Statistics and Probability Letters*, **28** (1996), 81-90.

[3] Duffie, Darrell, *Dynamic Asset Pricing Theory*, Princeton University Press, Princeton, New Jersey, 1992.

[4] Heynen, Ronald and Harry Kat, Partial barrier options, *Journal of Financial Engineering*, **3**, no.3/4 (1994), 253-274.

[5] Hull, John C., *Options, Futures, and Other Derivatives*, Upper Saddle River Prentice Hall, New Jersey, 1997.

[6] Iyengar, Satish, Hitting lines with two-dimensional Brownian motion, *SIAM Journal of Applied Mathematics*, **45**, no.6 (1985), 983-989.

[7] Karatzas, Ioannis and Steven Shreve, *Brownian Motion and Stochastic Calculus*, Springer, New York, 1997.

[8] Kunitomo, Naoto and Masayuki Ikeda, Pricing options with curved boundaries, *Mathematical Finance*, **4**, no. 2 (1992), 275-298.

[9] Musiela, Marek and Marek Rutkowski, *Martingale Methods in Financial Modelling*, Springer, Berlin, 1998.

[10] Revuz, Daniel and MartinYor, *Continuous Martingales and Brownian Motion*, Springer, New York, 1991.

[11] Rudin, Walter, *Principles of Mathematical Analysis*, McGraw Hill, Inc., New York, 1976.

[12] Shepp, Larry, Joint density of the maximum and its location for a Wiener process with drift, *Journal of Applied Probability*, **16**, (1979), 423-427.

## BIOGRAPHICAL SKETCH

David Brask was born in St. Louis, Missouri. He moved to south Florida where he attended school from fourth through twelfth grade. After graduating from Deerfield Beach High School in 1991 he went to the University of Florida. He was admitted into graduate school early via the Mathematics Department's accelerated master's degree program. David received his bachelor's degree in 1995 for a double major in mathematics and statistics. The following year he completed his master's degree in mathematics with a specialization in applied mathematics. After studying image compression and processing for about one year, his research moved into the field of mathematical finance. Soon after changing focus, David had an internship at William R. Hough & Co., an investment banking firm, to gain experience in the financial field. He was then certain that this was the area in which he wished to do his research.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Joseph Glover, Chairman
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Murali Rao
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

David Wilson
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Ira Horowitz
Graduate Research Professor of Decision and Information Sciences

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Mahendrarajah Nimalendran
Associate Professor of Finance, Insurance and Real Estate

This dissertation was submitted to the Graduate Faculty of the Department of Mathematics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

December 1999

Dean, Graduate School